Plasma Physics—Accelerators—Thermonuclear Research

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FOREWORD

PLASMA PHYSICS–ACCELERATORS–THERMONUCLEAR RESEARCH

(Journal of Nuclear Energy Part C)

IN the last few years one of the classical branches of physics, which had for some years been rather quiescent, has suddenly become fresh and exciting. Plasma physics is part of that study of gaseous discharges to which physics owes not only the fundamental discoveries of electrons and of isotopes but a great number of useful devices. Its renewed importance comes from the hope that it will be possible to derive controlled energy from fusion reactions between nuclei of isotopes of hydrogen, and so solve the problem of the supply of energy for all time. This involves heating the gas to temperatures of many million degrees at which it becomes a wholly ionized plasma of electrons and nuclei.

In view of the immense technical possibilities a large amount of work is now in progress in many nations, as was shown at the second Geneva Conference on the Peaceful Uses of Atomic Energy last August, where the thermonuclear section attracted great attention. It is desirable that papers on this subject, now scattered in a great many journals and published in many languages should be collected together in one journal. It is especially important that prompt and accurate translations of the numerous and important papers produced in Russia should be available for physicists unable to read the originals. These it is intended this journal should supply.

The design and construction of particle accelerators is another field of physics in which very large sums of money are now being spent. Here also a large proportion of the work is done in Government establishments or in the great international laboratory of C.E.R.N. It will be helpful to workers in this field also to have the literature collected. In fact there are several points of contact between this and the thermonuclear field more particularly on the side of electrical engineering, so they should be able to share a common journal harmoniously.

The new journal has a wide support from many countries, and will help to increase international collaboration over growing and important fields.

J. P. Thomson

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FOREWORD

A new and promising field of research and technological development has crystallized out of the ferment caused by the successful release of thermonuclear energy with the hydrogen bomb. The potential value to the world of sustained nuclear reactions which can provide controlled release of nuclear energy, based on fuel sources which are truly inexhaustible, provides an incentive which has already captured the imaginations of many of the world's best scientists and engineers.

The principles and techniques already emerging in the new technology of thermonuclear science are based primarily on the experience gained in two formerly independent fields, high energy accelerators and gaseous discharge. Accelerator technology has contributed significantly, particularly in the control and focusing of particle beams by electric and magnetic fields. Devices for studying plasma concentrations and pinch effects already involve shaped magnetic fields of unusually high flux density, which go far beyond either gas discharge or accelerator practice. Gas discharge phenomena are being pushed into a region of higher plasma densities than ever before studied, requiring new approaches in the engineering fields of heat removal and vacuum techniques.

This new journal has an important and useful function in sponsoring this merger of scientific disciplines and in publicizing developments in this growing field of technology.

M. Stanley Livingston

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J. Nucl. Energy, Part C: Plasma Physics, 1959, Vol. 1, pp. 3 to 7. Pergamon Press Ltd. Printed in Northern Ireland

SOME INVESTIGATIONS ON THE PHYSICS OF A HEATED PLASMA AND ITS CONTAINMENT

M. A. LEONTOVICH
U.S.S.R. Academy of Sciences

Abstract—The following theoretical and experimental investigations in plasma physics, performed at the Atomic Energy Institute of the U.S.S.R. Academy of Sciences were reported by Academician M. A. LEONTOVICH in a lecture given at the Atomic Energy Research Establishment, Harwell, England on April 28, 1959.

Some results of an experimental investigation of the currents in a plasma in toroidal chambers with a strong longitudinal magnetic field (V. S. MUKHOVATOV and V. S. STRELKOV)

THE stability condition for a plasma turn in a longitudinal magnetic field can be written down as follows*

$$H_z/H_\phi > R/a \qquad (1)$$

where H_z is the longitudinal magnetic field strength, H_ϕ is the discharge current field strength and R and a are respectively the radii of the torus and pinch cross-section. Expression (1) can be represented in the form

$$H_z/J = k\,2R/ca^2 \qquad (2)$$

where J is the discharge current and k is a coefficient characterizing the margin of stability.

Stability of pinches was studied experimentally in a toroidal assembly.† The discharge chamber is made of 0·1 mm thick stainless steel. The torus diameter is 125 cm and the diameter of the torus tube is 47 cm. Two diaphragms are mounted in the chamber. The diameter of the diaphragm aperture is 26 cm. The discharge chamber is located inside a copper torus with 2 cm thick walls. Quasi-stationary magnetic fields up to 10,000 oersted can be attained in the apparatus. The primary winding of the air transformer is energized by a condenser bank.

By varying the ratio H_z/J it can be observed clearly how the current is pinched towards the diaphragm apertures. The dependence of j_D/j_G on $k = \dfrac{H_z ca^2}{2JR}$ is depicted in Fig. 1 (j_D is the mean current density over the area covered by the diaphragm and j_G is the mean current density over the diaphragm aperature).

For $k \sim 1$, the mean current density j_D is about 30 per cent of the current density j_G. With increase of k this ratio rapidly drops and already for values $k = 3$ j_D/j_G does not exceed 7 per cent. Oscillograms of the current J and its derivative $\dot{J}$ for $k = 1{\cdot}1$ and $k = 3{\cdot}7$ are presented in Fig. 2. The amplitude of oscillation of the current derivative significantly decreases with increase of k i.e. of H_z/H_ϕ. Streak photographs of the plasma column illustrate firstly the decrease of the radius of the luminous region with increase of k and secondly a more uniform glow.

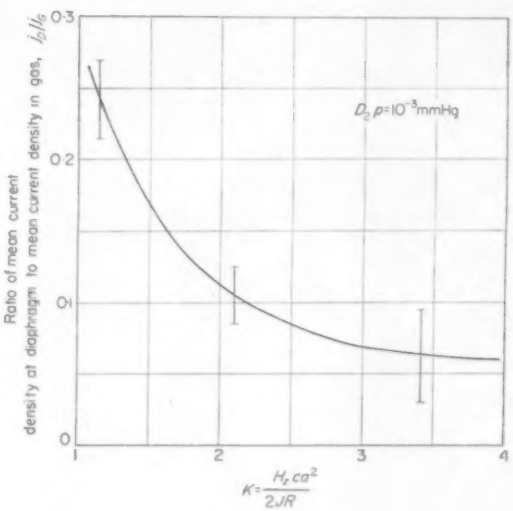

FIG. 1.

A possible interpretation of these facts is that for $k > 2$ the plasma column is restricted by the diaphragms and macroscopic oscillations for which $\lambda \sim R$ are not observed. Unfortunately due to the weak luminosity of the pinch in deuterium the photographs are not very distinct. Photographs of the pinch in argon are shown in Fig. 3; they are similar to those for deuterium. Oscillograms of the discharge voltage and discharge current and its derivative obtained under similar conditions are presented in Fig. 4.

* BRAGINSKY S. I. and SHAFRANOV V. D. (1958) *2nd Geneva Conference*, Report 2500.

† DOLGOV-SAVELIEV G. G. *et al.* (1958) *2nd Geneva Conference*, Report 2226.

Acknowledgements—The authors wish to express their thanks to L. A. ARTSIMOVICH and N. A. YAVLINSKY for guidance during the course of their experiments.

Some results of an experimental investigation of the plasma in a magnetic mirror trap (M. S. IOFFE, R. I. SOBOLYEV, V. G. TELKOVSKY, and E. E. YUSHMANOV)

The lifetime of a low density plasma in a magnetic mirror trap was studied. A hydrogen plasma of density of 10^9–10^{10} cm^{-3} and ion energy of 3–5 keV was obtained by accelerating ions from a cylindrical plasma column directed along the trap axis with a radial electric field.

A general description of the apparatus has been given by L. A. ARTSIMOVICH (1958)* and by E. E. YUSHMANOV (1958).†

Since, in the conditions of the experiment under consideration, the escape time of fast ions from the trap due to scattering on ions (and on residual neutral gas particles) was at least three orders of magnitude greater than the charge exchange time, it would be expected that charge exchange is practically the only source of ion losses, provided that no specific plasma mechanisms exist which lead to an even faster escape of the plasma from the trap.

The plasma lifetime was measured by recording the flux of fast neutral atoms due to charge exchange after switching off the radial electric field. For this purpose rectangular accelerating electric pulses of several tens of μsec duration were applied; the current of secondary electrons ejected by the fast atoms from an insulated electrode located near the wall in the middle of the trap was recorded on an oscillograph. A typical oscillogram obtained in such measurements is shown in Fig. 5, the secondary electron current is depicted in the upper part and the accelerating voltage pulse v in the lower part of the figure.

The lifetime τ was taken to be the time required for the current to decrease by a factor of e (at a given pressure the current due to fast atoms and the electrons ejected by them is proportional to the ion density).

The dependence of τ on hydrogen pressure in the trap was measured in the pressure range from 5×10^{-7} to 2×10^{-5} mm Hg for several values of the magnetic field strength H_0. Results of the measurements are represented in Fig. 6. They indicate a linear dependence of $1/\tau$ on pressure which is precisely what one would expect in the case of charge exchange; however the straight lines drawn through the experimental

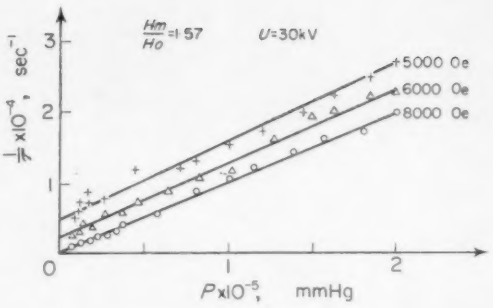

FIG. 6.

points upon extrapolation to zero pressure intersect the ordinate axis above zero, at $1/\tau_n$, this point being the higher the weaker the magnetic field.

Pressure measurement errors cannot explain this fact; it apparently signifies that under the conditions of the present experiment, besides charge exchange, there is an additional ion escape mechanism with a characteristic time τ_n.

For $H_0 = 8000$, 6000 and 5000 oersted the values of τ_n are respectively 1500, 400 and 200 μsec (the mirror ratio being $H_m/H_0 = 1{\cdot}57$).

Similar measurements of the dependence of τ on pressure at various mirror ratios lead to a strong dependence of τ_n on H_m/H_0. Thus, for $H_0 = 8000$ oersted and $H_m/H_0 = 1{\cdot}57$; $1{\cdot}45$; $1{\cdot}25$ the values of τ_n are respectively 1500, 750 and 200 μsec.

Acknowledgement—The authors wish to express their thanks to L. A. ARTSIMOVICH for valuable discussions.

On the stability condition for a low pressure plasma (B. B. KADOMTSEV)

Within the framework of the hydrodynamical theory the stability condition has been derived for a low pressure plasma $\left(\varepsilon = \dfrac{8\pi p}{H^2} \ll 1\right)$. This condition permits one to determine the minimal restrictions on a magnetic field which should be capable of confining the plasma. For fields with closed lines of force the stability condition was known earlier (KADOMTSEV, 1958).‡

In a paper completed recently by the author a study was made of the stability of low pressure plasmas in systems with unclosed field lines but of a topologically toroidal type, that is, in which constant pressure surfaces consist of nested toroidal surfaces. The energy principle underlies the analysis. It is shown that for the most dangerous disturbances which vary slowly along the field lines and for systems with

* ARTSIMOVICH L. A. (1958) *Second Geneva Conference*, Report 2293.

† YUSHMANOV E. E. (1958) *Plasma Physics*, Vol. 4, p. 235, Pergamon Press, London.

‡ KADOMTSEV B. B. (1958) *Plasma Physics* **4**, 16; *Progress in Nuclear Energy, Series* XI, Vol. 1, pp. 565–575. Pergamon Press, London.

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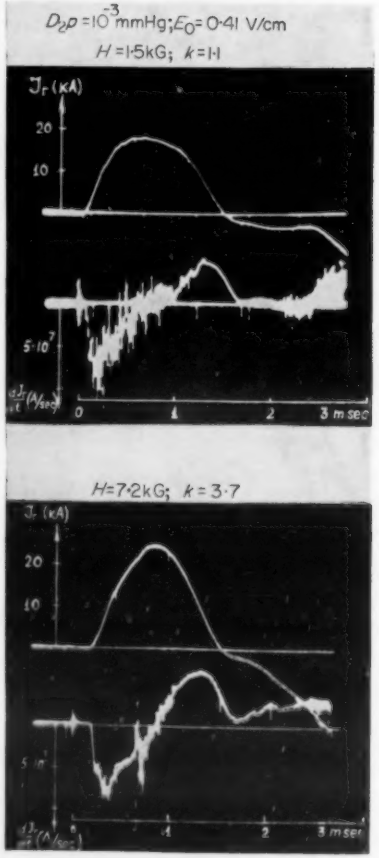

FIG. 2.

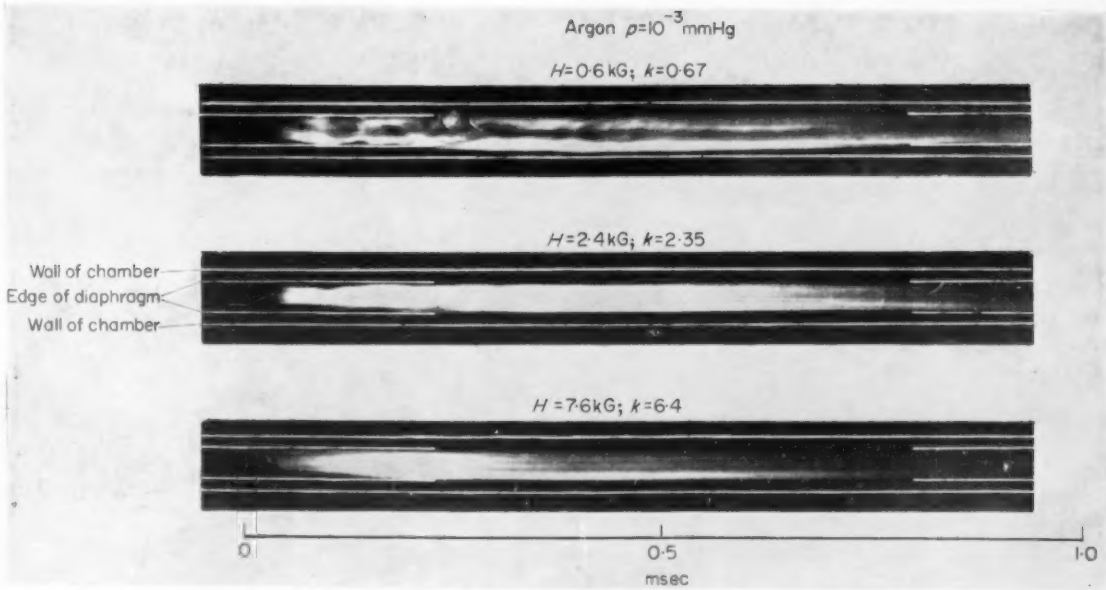

FIG. 3

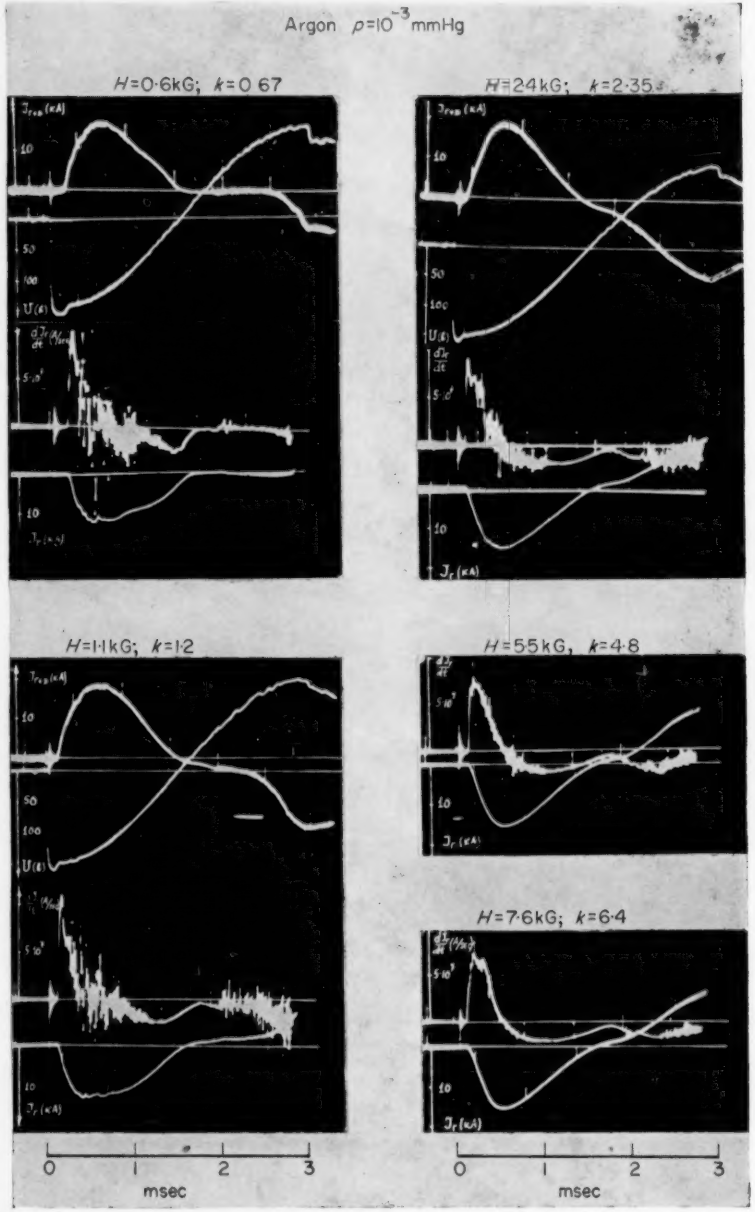
Argon $p=10^{-3}$ mmHg
H=0·6kG; k=0 67
H=24kG; k=2·35
H=1·1kG; k=1·2
H=5·5kG, k=4·8
H=7·6kG; k=6·4
msec
msec

FIG. 4.

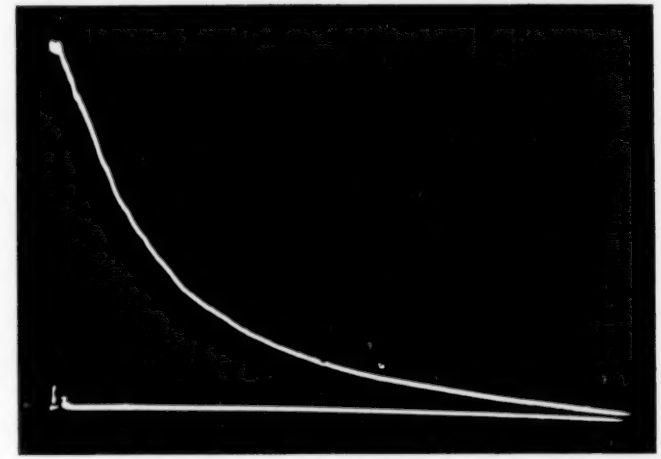

FIG. 5.

sufficient field inhomogeneity the necessary stability condition is given by the relation

$$\nabla p \nabla U < (\nabla \mu)^2 \pi a^2 / |4U| \qquad (1)$$

where a is a mean value of the small radius of a toroidal surface of prescribed pressure p; U is the mean value of the integral $\int dl/H$ along the field line over the entire length of the toroid and μ is the number of minor radius revolutions of the field during one complete circuit of the toroid.

For a weakly inhomogeneous field such that $|\nabla U/U|$ is a small quantity of the order of ε, the stability condition should be modified so as to take into account additional small terms; the stability condition then turns out to be somewhat more complicated and, for example, in the case of a straight pinch with cylindrical symmetry it is identical with the Suydam criterion (SUYDAM, 1958).*

In the derivation of condition (I) no restrictions were imposed on the field geometry excepting that of toroidality, and therefore it is more general than the analogous criterion for stability derived by JOHNSON *et al.* (1958)† for a straight weakly inhomogeneous magnetic field.

On the interchange instability for the cylindrically symmetrical plasma column (B. B. KADOMTSEV)

Convective or interchange instability was studied for the simplest axially symmetrical configuration of a plasma column confined by the field of an axial current; no restrictions were imposed on the plasma pressure (quantity $\varepsilon = 8\pi p/H^2$).

A necessary and sufficient stability condition can easily be obtained in the hydrodynamic approximation in the absence of dissipative processes. It can be expressed as follows

$$-\frac{d\ln p}{d\ln r} < \begin{cases} 1/\varepsilon & \text{if} \quad \varepsilon > 2/3\gamma \\ \dfrac{4\gamma}{2+\gamma\varepsilon} & \text{if} \quad \varepsilon < \dfrac{2}{3\gamma} \end{cases} \qquad (1)$$

where γ is the adiabatic exponent, equal to 5/3 for an ideal gas and $\varepsilon = 8\pi p/H^2$.

According to the relation (1) the plasma pressure should not decrease too rapidly with the radius. It follows from this condition that stable configurations should have a singularity at $r \to 0$ and apparently can be realized experimentally only in the presence of a rigid current carrying conductor at the centre of the column.

The stability of the plasma column was also investigated in the same work with the aid of the Boltzmann equation. It was found that in the absence of collisions the stability condition is somewhat less stringent than (1). However there exists a second, a collisional branch of oscillations with frequencies of the order of the collision frequency between particles and taking account of this branch leads exactly to criterion (1).

Stability of entropy waves i.e., of column disturbances for which the pressure at each given point remains constant, was also studied in the paper under discussion in the hydrodynamical approximation. Electron and ion drift leads to propagation of these waves along the column axis with the drift velocity and, under certain conditions, to build-up of the waves with an increment of the order of $\omega \sim V_d \cdot k$ where V_d is the drift velocity, $V_d \sim V_T \frac{r_H}{r}$, r_H the Larmor radius, V_T the thermal velocity and k the wave number.

The stability condition for these oscillations is

$$\frac{d\ln T}{d\ln r} < 1 + \left(\frac{7}{10} + \frac{\varepsilon}{4}\right)\frac{d\ln p}{d\ln r} \qquad (2)$$

For a small curvature ($r \to \infty$) it goes over to the condition

$$\frac{d\ln T}{d\ln p} < \frac{7}{10} + \frac{\varepsilon}{4},$$

obtained previously by TSERKOVNIKOV which is the hydrodynamical analogue of one of the criteria deduced by him with aid of the Boltzmann equation (TSERKOVNIKOV, 1957).‡

The physical cause of entropy wave instability is the same as that of convective instability, namely, the tendency of the plasma to expand. Excitation of entropy waves is a consequence of non-adiabatic convection in which heat exchange occurs between various parts of the plasma.

On some results of the study of the stability of plasma of axial symmetry by means of kinetic equations (L. I. RUDAKOV and R. Z. SAGDEYEV)

A collision-free kinetic equation was used to study the build-up of oscillations in the problem of the same symmetry. It was demonstrated by the authors in this way that entropy as well as magneto-acoustic waves can be built up. The condition of stability

* SUYDAM B. (1958) *2nd Geneva Conference*. Report 354. United Nations, New York.

† JOHNSON J., OBERMAN C., KULSVUD R. and EIEMAN E. (1958) *2nd Geneva Conference*, Report 1875.

‡ TSERKOVNIKOV (1957) *Zh. eksp. teor. Fiz.* **32**, 67.

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with respect to magneto-acoustic oscillations has the form

$$\left(1-\frac{L}{r_H}\frac{V_A}{V_T}\right)\frac{d\ln T}{d\ln H}<\frac{L}{r_H}\frac{V_A}{V_T}+\frac{H^2}{8\pi nT}\quad \frac{1}{L}=\frac{dH/dr}{H}$$

where

$$V_A=\frac{H}{\sqrt{4\pi nm}}\quad V_T=\sqrt{\frac{2T}{m}}\quad r_H=\frac{V_T}{\omega_H}$$

The analysis showed that the physical cause of instability of these oscillations as well as of the drift waves is the build-up of oscillations by particles drifting with a velocity close to the phase velocity of the wave. These particles experience a systematic drift along the gradient of the unperturbed magnetic field and impart energy to the wave. Thus, even in the absence of collisions, excitation of long wave oscillations (wave length larger than the Larmor radius) is due to movement of the plasma across the magnetic field (as in convection) but only those particles which fall into resonance with the wave participate in this motion.

On the influence of finite conductivity on the stability of a pinch in a strong longitudinal magnetic field (V. D. SHAFRANOV)

In recent experiments performed with straight chambers and with toroidal chambers with a strong longitudinal magnetic field (H_z) satisfying the condition $H_z/H_\varphi > R/a$ it has been possible to obtain a pinch possessing a certain degree of stability. However, in all experiments the plasma conductivity was not large and did not exceed 10^{15} CGS units.

On the other hand, previous investigations of the stability of plasma systems with surface as well as distributed currents were carried out on the assumption that for small oscillations the plasma conductivity can be considered to be infinite, i.e. during the oscillations the magnetic field is 'frozen' in the matter and diffusion is absent.

An attempt has been made to study the stability of a pinch stabilized by a longitudinal magnetic field by taking into consideration finite conductivity and hence diffusion of the plasma. Influence of diffusion of the stabilizing longitudinal field H_z would seem to be of greatest interest, whereas diffusion of the current field, H_φ, as one can see readily from qualitative considerations, should not change the general picture appreciably. Investigation of stability therefore was carried out for the simplified model of a pinch with a surface axial current; the perturbed magnetic field of this current is determined by assuming that $\sigma=\infty$; stabilization is achieved with a strong uniform field $H_z \gg H_\varphi$ the diffusion of which is taken into account.

The calculations show that when finite conductivity of the plasma is taken into account the 'stabilized' pinch proves to be unstable, the perturbations increasing according to the law $\exp(\omega t)$ where the increment ω is defined by the relations

$$\begin{cases}\omega^3=k^2\dfrac{H_\phi^2}{4\pi\rho}\dfrac{c^2}{4\pi\sigma a^2}\dfrac{H_\phi^2}{H_z^2}\ (m=0)\\ \omega^3=\dfrac{H_\phi^2}{4\pi\rho a^2}\dfrac{c^2}{4\pi\sigma a^2}\left(m=1,\ \dfrac{H_\phi}{H_z}<ka\ll 1\right)\\ \omega^3=k^2\dfrac{H_\phi^2}{4\pi\rho}\dfrac{c^2}{4\pi\sigma a^2}\ (m=1\ ka\ll 1)\end{cases}$$

Roughly, $\omega^3=\omega_0^2\cdot 1/t_{\text{dif}}$ where $t_{\text{dif}}=\dfrac{4\pi\sigma a^2}{c^2}\dfrac{H_z^2}{8\pi p}$ is a characteristic diffusion time for a pinch of radius a and $1/\omega_0$ is the time of instability growth in the case of poor conductivity; in the presence of kinks it is H_z/H_φ times smaller than for axially symmetrical perturbations.

The rate of development of instabilities is found to be very large. However these perturbations are of a local nature. The plasma flows from the disturbed areas along field lines in a thin surface layer. Therefore it may be expected that these perturbations will be less dangerous than perturbations of the unstabilized pinch.

Role of impurity radiation in the energy balance of the pinch (V. I. KOGAN)

One of the modes of energy loss in a hydrogen or deuterium plasma is the radiation due to the presence of impurities. This radiation primarily leads to a decrease of the electron temperature.

The problem was considered in the following simplest form. A completely ionized hydrogen pinch is confined mainly by the magnetic field of its own current. The ion and electron temperatures are assumed to be the same (at least in order of magnitude). Furthermore, it is assumed that a prescribed amount of a given impurity is present in the plasma such that the total number of electrons in the shells of the impurity atoms is smaller than the number of electrons from the hydrogen atoms. In the energy balance account is taken of Joule heating of the plasma and its cooling as a result of emission of radiation from the impurity as well as hydrogen ions in free-free, free-bound and bound-bound transitions. It was found that if the amount of impurities exceeded $\sim$(0·01 — 0·1) per cent most of the radiation losses were due to the impurity ions.

Under the assumptions made above, the ratio of

impurity radiation losses to Joule heating assumes the following simple form

$$\frac{Q_{\text{rad}}}{Q_{\text{Joule}}} = \frac{N^*}{N^*_{\text{crit}}} \quad N^*_{\text{crit}} = \frac{\text{constant}}{T^{\frac{1}{2}} F_z(T)}$$

where N^* is the total number of impurity atoms per unit length of the pinch; $F_z(T)$ is the total intensity of bremsstrahlung, recombination and line radiation emitted by an impurity of atomic number Z per unit volume, the electron and impurity densities being assumed to equal unity. The quantity $F_z(T)$ was evaluated with aid of the results obtained by KNORR* under the assumption that the effective ionization depth of the impurity particles (which is important for the probabilities of the radiation processes) is defined, as a function of temperature, by the balance of two processes: of electron impact ionization and radiative recombination. This assumption is correct (at least qualitatively) for 'slow' discharges whose duration is much greater than the effective ionization time.

The function $F_z(T)$ thus obtained is such that N^*_{crit} turns out to be practically constant in order of magnitude over a broad range of temperature (from 10 to 10^4 eV). Thus

$$\text{for oxygen} \quad (Z = 8) \quad N^*_{\text{crit}} \sim 10^{16}\ \text{cm}^{-1},$$
$$\text{for calcium} \quad (Z = 20) \quad N^*_{\text{crit}} \sim 10^{15}\ \text{cm}^{-1}.$$

Therefore, under the assumptions made above, the fraction of energy carried off by the impurity radiation practically depends only on the absolute number of impurity atoms in the cross-section of the pinch.

This means that if the percentage of impurities is low they can be of importance in heat loss only if the total number of particles in the cross-section is large ($>10^{17}$ cm^{-1}). If, on the other hand, the number of hydrogen ions in the cross-section is lower than ($10^{16} - 10^{17}$) cm^{-1} a low percentage of impurities should ensure that the latter play an insignificant role in the energy balance.

Motion of particles in magnetic traps (A. I. MOROZOV and L. S. SOLOVYOV)

At present an urgent task is the development of a kinetic theory of equilibrium and stability of plasma configurations. A natural preliminary step in this direction would be to examine in detail the motion of individual particles in magnetic traps. A study of this type was undertaken in the drift approximation. For fields possessing translational, axial and helical symmetries exact integrals of the drift equations have been obtained for $\vec{H}\,\text{curl}\,\vec{H} = 0$. The integral for an axially symmetric magnetic field has been applied to analyse the motion of particles in the field of a current-carrying plasma turn located in a longitudinal magnetic field. For a thin torus the largest displacement of the particle drift trajectory due to toroidality is $\sim r_H \sqrt{a/R}$ where r_H is the Larmor radius in the field of the current-carrying ring, a is the distance from the ring, and R is the torus radius.

An analysis of the motion of particles in corrugated toroids revealed that, in principle, escape of the particles from the traps can be prevented. The maximal orbit shift due to toroidality in this case is

$$\frac{8}{\alpha}\left(\frac{H_0^2}{2\alpha R h^2}\right)^{1/3} \quad \text{where} \quad \alpha = \frac{2\pi}{L},$$

is the period of the corrugations, H_z and h are respectively the constant and variable components of the longitudinal fields. The shift does not depend on the magnitude of the Larmor radius.

The motion of particles in a magnetic field produced by a helical coil in the presence of a longitudinal field has also been considered. In a trap of this type the lines of force form magnetic surfaces which enclose the ring axis of the toroid $r = R$. Beyond a certain magnetic surface (separatrix), the lines of force leave the system. If the particles are located inside the separatrix they will remain there for arbitrary values of the ratio $V_\perp/V_\parallel$ provided the values of R are sufficiently large and the rotational transform angles are sufficiently small. Thus, in principle, a field of this type can equally be employed to produce an ideal trap. However in practice it should be difficult to create an 'absolute' trap of this type since groups of particles exist for which the displacements due to toroidality are very large. These are particles for which mean velocities of revolution about the ring axis are close to zero.

Expressions are presented in the paper which are also valid in the presence of a potential electric field; however, specific trajectories have been determined only in the absence of an electric field.

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* KNORR (1958) *Z. Naturf.* **13a**, 941.

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PRODUCTION OF HIGH DENSITY PROTON CURRENTS FOR COLLIDING BEAM ACCELERATORS

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Abstract—A head-on collision of two 15 GeV protons produces an energy in the centre of mass equivalent to about 500 GeV with the target at rest. To achieve significant production of secondaries, intense beams of particles are required, and this is obtained by 'stacking' successive pulses of particles at high energy. Limitations imposed by the accelerating system are examined and upper limits are given to the densities that can be obtained under different assumptions.

1. INTRODUCTION

AT the C.E.R.N. accelerator conference in 1956, KERST (KERST *et al.*, 1956) and other members of the M.U.R.A.* group outlined the possibilities of using fixed field alternating gradient (F.F.A.G.) magnet rings for colliding beam accelerators. In this accelerating system, two beams of high energy particles are directed against each other giving energies in the centre of mass system very much higher than can be produced by a single beam hitting a stationary target. High beam densities are needed for sufficient interactions to take place and, in order to achieve these densities, successive pulses of particles must be stacked at high energy in the d.c. field. Two of these stacked beams would circulate in opposite directions and interact in a target section (or sections), any one particle passing many times through the target section. Losses from the beams due to interactions and collisions with the residual gas molecules would be made up continually by further pulses of particles. The theory of the 'stacking' process was given by SYMON and SESSLER (1956) at the C.E.R.N. conference and experiments have since confirmed this theory (TERWILLIGER *et al.*, 1957a).

Since the C.E.R.N. conference, extensive studies have been made of the dynamics in both radial sector and spiral-ridge F.F.A.G. fields. In finding possible designs, many instabilities caused by both intrinsic and error resonances have to be avoided. In addition, with spiral ridge F.F.A.G., the insertion of straight sections introduces non-scaling features which have proved troublesome, although possible solutions exist. For colliding beams, the OHKAWA (1958) radial sector machine is probably of greatest interest since two beams can circulate in opposite directions in the same magnet.

We shall assume in this paper that one or other of these designs has been found entirely feasible, and shall discuss the possible beams that might be produced for a colliding beam accelerator and the r.f. systems that would be needed to provide them. In the present treatment, many simplifications have been made, but it will be shown that, even for the idealized system, there are definite limitations and requirements to be met. Finally a system will be given which satisfies these requirements and for which further more detailed calculations could be made.

2. THE REQUIREMENTS FOR A SYNCHROCRASH BEAM

We shall be concerned with the 'crashing' of two 15 GeV proton beams; this is equivalent to a single beam of 500 GeV hitting a stationary target. To achieve a reasonable yield of interactions each of these beams should contain of the order of 10^{15} particles (KERST *et al.*, 1956) with a radial spread (due to energy spread) of only a few centimetres. Now the maximum circulating current that can be obtained at full energy is equal to

(a) the number, n_p, of particles injected per pulse, multiplied by

(b) the number, n_s, of pulses per second introduced into the machine multiplied by

(c) the circulating time, T_c, (sec) before the particles at full energy are lost through interactions, gas scattering or otherwise.

The purpose of this paper is to describe the limitations which must be imposed on the system, to ensure that the particles are not lost during acceleration and that the final energy spread is within prescribed limits. We shall show that an upper limit to the circulating current is likely to be of the order

$$10^{12} \times 10^2 \times 10^2$$

where the numbers refer to (a), (b) and (c) above.

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(a) The first of these factors is equivalent to an injection current of 30 mA for one turn at 40 MeV, and also corresponds roughly to the space charge limit in a spiral ridge machine. If multi-turn injection is possible (as may well be the case in an OHKAWA machine) the first factor may be put up to 10^{13}, by, for example, injecting 10 turns from a 30 mA injector (60 turns from 5 mA injector). With the larger stable (betatron) area available in an OHKAWA machine, this current of 10^{13} particles should be within the space charge limit (see Section 5.2).

(b) The product of the first two factors gives the mean current which has to be accelerated to 15 GeV. For a mean current of 10^{14} particles per second there will be 240 kW in each beam, that is roughly 0·5 MW beam loading. An order of magnitude increase in this is not realistic.

(c) From a consideration of the interactions of the beam with the residual gas molecules it is found that a circulating time of 100 sec corresponds to a gas pressure of 10^{-5} mm. This time might possibly be increased to 1000 sec at a pressure of 10^{-6} mm. Unfortunately, we should not then be able to increase the circulating current to

$$10^{12} \times 10^2 \times 10^3.$$

This would mean stacking 10^5 pulses at high energy and we shall show that it is impossible to do this without giving the final beam an excessive energy spread, unless the repetition rate at injection is reduced from 100 pulses per second.

If a circulating time of 1000 sec is achieved an alternative to the above case would be

$$10^{12} \times 10 \times 10^3.$$

Here again the total number of stacked pulses is $10 \times 10^3 = 10^4$. This method has the advantage that the smaller mean current may be accelerated with a much simpler r.f. system.

If it is possible to inject 10^{13} particles per bucket then two possible schemes would be

$$10^{13} \times 10 \times 10^2$$

or $$10^{13} \times 10 \times 10^3.$$

Clearly, it is desirable to inject as many particles per bucket as is possible.

It is important to note that the maximum of 10^{16} (10^{17} if multi-turn injection is possible) is based on very idealistic assumptions, and the tolerances on achieving this are very severe.

3. BEAM STACKING

In their C.E.R.N. Symposium paper, SYMON and SESSLER (1956) outlined several possible systems of acceleration for F.F.A.G. machines. Since then, the discovery of the phenomenon of r.f. 'knockout' (TERWILLIGER, 1957b) (see Section 4) has greatly increased the difficulties associated with some of these schemes. The simplest and most feasible scheme seems to be that of beam stacking, and this is the only scheme with which we shall be concerned in this report. Here again r.f. 'knockout' considerations are very important and impose certain definite limitations on possible systems (see again Section 4).

In the beam stacking scheme, the acceleration of particles is accomplished in a number of stages, each with an independent r.f. system. In the first stage, pulses of particles are accelerated from the injection energy to the first stacking energy, the second stage accelerates the particles from the first stacking level to the second and so on to the full energy where the particles are stacked to form the synchrocrash beam.

Let us consider a typical stage taking a particle from an energy E_1 to E_2. This stage will consist of a number of cavities spaced round the machine, all working at the same angular frequency ω_{rf}, although possibly at different phases. The frequency, ω_{rf}, will sweep out some harmonic h of the angular frequency range associated with the rotation of the particles $\omega(E_1) - \omega(E_2)$. Successive pulses of particles will be accelerated from E_1 to E_2, where they will be left to circulate while further pulses are brought up. The cavities will physically occupy only that part of the aperture appropriate to their energy range (see Table 1) although there will of course be overlapping at the stacking levels.

The equations of motion are

$$\left.\begin{aligned} \frac{dE}{dt} &= \frac{eV(t)}{2\pi}\,\omega(E)\sin\phi \\ \frac{d\phi}{dt} &= h\omega(E) - \omega_{rf}(t) \end{aligned}\right\} \qquad (1)$$

where $\phi = h\theta - \int_0^t \omega_{rf}(t)\,dt$

These equations can be put in canonical form by introducing SYMON's variable

$$W(E) = \int_{E_1}^{E} \frac{dE}{\omega(E)}.$$

The quantities W and ϕ are then canonical variables, and we may now discuss the system in terms of canonical phase area and apply Liouville's theorem.

This theorem states that density in phase space measured with the particles is conserved. We shall, of course, in general be concerned with densities averaged over sizeable regions of phase space, and care must be taken to see that very little 'empty' phase space is mixed into these regions.

This 'mixing' may occur in the following ways:—

(1) At the beginning of an accelerating cycle, the r.f. voltage is zero and the particles are spread out over 2π in phase ϕ, and over a certain energy range ΔE_1. These particles must be 'condensed' into an accelerating bucket and, in order to avoid mixing of phase space, the voltage V must be raised adiabatically (see Section 5.5) to $V_{\max}$, the accelerating voltage. This adiabatic switch-on will take a certain time S which must be allowed for in calculating the duty cycle, and this imposes severe restrictions on the duty cycle at injection. We shall assume that, provided this time is allowed for, no mixing takes place.

(2) R.F. jitter, (sudden changes in ω_{rf} during acceleration) will also mix in empty phase space. We shall, however, ignore this effect.

(3) When subsequent pulses (buckets) are brought up to E_2, they will affect the pulses already stacked at E_2. The particles already at E_2 will be "phase displaced" (SYMON and SESSLER, 1956) downwards in energy to make phase room for the new particles. Inevitably, some phase mixing will occur but we shall assume that it may be ignored, provided we ensure that the incoming bucket is full of particles (i.e. that the bucket itself does not bring in any empty phase space).

To determine the parameters associated with a bucket, we use the method of BOHM and FOLDY (1946). Defining a synchronous energy E_s by the relation $h\omega(E_s) = \omega_{\text{rf}}$ and a synchronous phase angle ϕ_s by

$$\frac{dE_s}{dt} = \frac{eV}{2\pi}\omega_s \sin\phi_s$$

we may write $E = E_s + \Delta E$ and linearize in ΔE. We obtain the usual BOHM and FOLDY phase equation.

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega}\right) = \frac{d}{dt}\left(\frac{E_s}{hK\omega_s^2}\dot{\phi}\right) = \frac{eV}{2\pi}(\sin\phi - \sin\phi_s) \quad (2)$$

where $K = \dfrac{E}{\omega}\dfrac{d\omega}{dE}$.

The bucket passing between E_1 and E_2 is characterized by the values of V and $\gamma = \sin\phi_s$. It will contain a canonical phase area given by

$$A_B = \frac{8}{\omega_s}\left\{\frac{2eV \cdot E_s}{\pi hK}\right\}^{1/2}\alpha(\gamma) \quad (3)$$

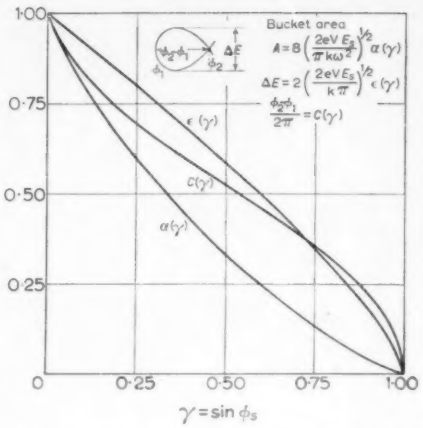

FIG. 1.—The area and widths of buckets for different $\gamma = \sin\phi_s$.

where $\alpha(\gamma)$ is shown in Fig. 1 and will take a time T to accelerate each pulse of particles

$$T = \frac{2\pi}{eV \cdot \gamma}\int_{E_1}^{E_2}\frac{dE}{\omega(E)} + S \quad (4)$$

(where S is the adiabatic switching time).

In what follows, we shall describe systems in terms of pulses per second. This will always refer to the repetition rate at input. After the first stage, a number of pulses may be accumulated by stacking so that the repetition rate is in general less for these stages. Nevertheless, the mean current accelerated (provided nothing is lost) is given by the number of particles per pulse, times the number of pulses per second at input. The first stage is therefore a very critical one, since, in order to get a high mean current, we require a high repetition rate and, consequently, a short acceleration time in this stage. This in turn restricts the time for adiabatic switch on. On the other hand, even if the injected particles were mono-energetic, an energy spread is inevitably induced by switch-on. It is important therefore to relate induced phase area to switch-on times. Before doing this, we shall begin with a brief discussion of r.f. knockout since this imposes important restrictions on the r.f. system and harmonic number.

4. R.F. KNOCKOUT

The particles in a stacked beam will circulate with some angular frequency ω_{st}. As further pulses are accelerated, the r.f. frequency will sweep through values out of resonance with ω_{st}, and the energy of the stack will be modulated slightly. This modulation is not normally important. However, under certain conditions, there can be resonance between this modulation and the betatron oscillations of the beam.

This r.f. knockout phenomenon causes growth of the amplitude of the betatron oscillations.

The first-order resonances may be represented by the equation

$$\frac{d^2\rho}{dt^2} + Q^2\rho = \frac{\Delta p}{p} \tag{5}$$

the principal terms of the R.H.S. being proportional to $\exp i(s\theta + nN\theta - \omega_{\mathrm{rf}}\, t)$. The L.H.S. represents the smoothed out betatron oscillation and Q is either the radial or vertical betatron frequency. The R.H.S. represents the energy modulation, N is the number of cavities and s (harmonic number) takes account of any linear phase shift between the cavities. The $nN\theta$ term must be replaced by $n\theta$ if the cavities are not placed symmetrically around the machine, or if there are errors in the voltage

We may write

$$\omega_{\mathrm{rf}}\, t = \frac{\omega_{\mathrm{rf}}}{\omega_{\mathrm{st}}}\,\theta = s\,\frac{\omega_p}{\omega_{\mathrm{st}}}\,\theta \tag{6}$$

where ω_p is the angular frequency of particles in the bucket being accelerated. Hence knockout frequencies occur when

$$Q = s\left(1 - \frac{\omega_p}{\omega_{\mathrm{st}}}\right) \pm nN$$

that is

$$\frac{Q}{N} = \frac{s}{N}\left(1 - \frac{\omega_p}{\omega_{\mathrm{st}}}\right) \pm n \tag{7}$$

The resonance diagrams are shown for $s = \frac{N}{2}$, N, $\frac{3N}{2}$ in Figs. 2, 3, 4. As s increases the range of $x = \frac{\omega_p}{\omega_{\mathrm{st}}}$ between resonance lines decreases, and thus we would choose to work with $S = N/2$, that is, to work the cavities in antiphase. To avoid knockout resonances, the best mode of operation is the fundamental with N, the number of cavities, large. In this case, the resonance diagram opens out to give a much larger trouble free range for $\frac{\omega_p}{\omega_{\mathrm{st}}}$.

From Fig. 2, we see that if Q/N is approximately equal to $n + 1/2$ where n is an integer, the allowable range of $\frac{\omega_p}{\omega_{\mathrm{st}}}$ is from 0 to 2, that is the two sets of cavities on either side of the stacked beam may take the particle from a very small frequency to nearly twice the stacked beam frequency without causing knockout. This gives very reasonable radial separation between stacking levels (see Table 1). We shall take h, the harmonic number equal to 10 in all further calculations

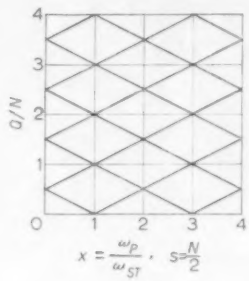

FIG. 2.—Resonance diagrams for r.f. knockout.

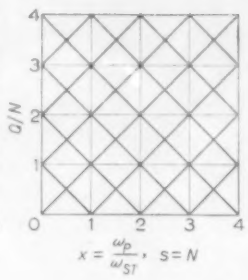

FIG. 3.—Resonance diagrams for r.f. knockout.

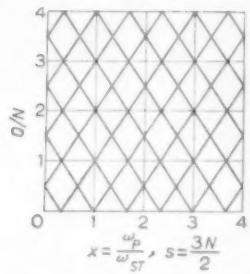

FIG. 4.—Resonance diagrams for r.f. knockout.

and ignore any further knockout resonances caused by non-linearities or errors.

5. THE FIRST STAGE

We shall now discuss in some detail the important first stage of the acceleration. To begin with, the number of particles that we can reasonably expect to be injected per pulse will be calculated. Then the relation between the phase area per pulse and the duty cycle at injection will be considered. Finally, the limitations imposed on the whole system by the first stage will be discussed.

5.1 *Injection*

The number of particles injected per turn

$$= \frac{2\pi R}{\beta \mathrm{ec}}\, I$$

where I is the mean current from the injector and R is the radius of the machine.

Thus for a 75 m radius machine, particles per turn

$$= \frac{2\pi \times 75 \times 10^{-3}}{1{\cdot}6 \times 10^{-19} \times 3 \times 10^{8}}\,\frac{I(\mathrm{mA})}{\beta}$$

$$= \frac{10^{10}}{\beta}\ /\mathrm{mA\ turn\ (or\ twice\ this\ for\ a\ 150\ m\ radius)}$$

With single turn injection at 40 MeV ($\beta = 0{\cdot}283$) from a linac, $3{\cdot}5 \times 10^{10}$ particles per mA mean current

would be injected. Thus a 30 mA injector would give 10^{12} particles per pulse.

5.2 *Space charge*

We may make some rough estimates of the effects of space charge forces on the particle dynamics by calculating the electric and magnetic fields inside the beam, assumed to have uniform charge density and to be moving in a straight line, and putting these into the betatron equations of motion. The main effect of these fields is to modify the betatron frequency. The smoothed-out equation of motion becomes

$$\rho'' + \left\{Q^2 - \frac{NRr_{\mathrm{pr}}}{4\pi \,.\, \Delta A}\frac{(1-\beta^2)^{3/2}}{\beta^2}\right\}\rho = 0 \qquad (8)$$

where N = total number of particles in the pulse
R = radius of the machine
r_{pr} = 'classical' radius of the proton
ΔA = cross-sectional area of the beam
Q = refers to either the radial or the vertical betatron frequency.

Thus

$$Q'^2 - Q^2 = (Q + Q')(Q' - Q) = \frac{NRr_{\mathrm{pr}}}{4\pi\Delta A}\frac{(1-\beta^2)^{3/2}}{\beta^2} \qquad (9)$$

where Q' is the modified betatron frequency.

$$\therefore \qquad Q' - Q \approx \frac{1}{2Q}\frac{NRr_{\mathrm{pr}}}{4\pi\Delta A}\frac{(1-\beta^2)^{3/2}}{\beta^2} \qquad (10)$$

We shall certainly require that

$$Q' - Q < 1/2$$

and therefore

$$N < \frac{4\pi\Delta A \,.\, Q}{Rr_{\mathrm{pr}}}\frac{\beta^2}{(1-\beta^2)^{3/2}}$$

At 40 MeV ($\beta = 0{\cdot}283$), $r_{\mathrm{pr}} = \dfrac{2{\cdot}8 \times 10^{-13}}{1837}$ and putting $R = 75 \times 10^2$ cm, we obtain

$$N < 1{\cdot}0 \times 10^{12}\,\Delta A \,.\, Q \qquad (11)$$

where ΔA is in cm^2. This should be reduced by half for a 150 m machine. Taking ΔA as greater than 3 cm^2 and Q greater than 3·5, we have that $\Delta A \,.\, Q > 10$. The particles, however, will be bunched in phase when they are accelerated and may thus occupy as little as one third of the circumference. As β increases, these space charge effects become smaller through the factor $\dfrac{\beta^2}{(1-\beta^2)^{3/2}}$ and as β approaches unity become negligible.

For the moment, we shall take 10^{12} particles as the pulse size, and, as we do not intend to stack many pulses together at low energy, space charge effects will be omitted from further calculations.

5.3 *Phase space area per pulse*

For the present, we suppose that the injected particles are mono-energetic. They then have no area in phase space but occupy a line of length 2π in phase. If the r.f. is switched on suddenly, phase oscillations will be induced immediately. Although the phase 'area' is still a line in accordance with Liouville's theorem, this line will soon spread itself over the effective area of the bucket because of the non-linearity in the phase oscillation frequency. To prevent this loss of density, the r.f. voltage must be switched on slowly. The phase oscillations will then damp and the resultant energy spread can be kept small.

The maximum area which can be tolerated is determined by the total number of pulses which have to be accommodated at full energy within a given energy range. At high energy, the relation between energy spread and radial spread for a 15 GeV machine with a 4 m radial aperture is approximately

$$\Delta R\,(\mathrm{cm}) \sim 7\Delta E\,(\mathrm{GeV})$$

Defining the total phase area as

$$A_T = \frac{\Delta E}{\omega} \times 2\pi \quad (\Delta E \text{ in MeV}, \omega \text{ in rad}/\mu\text{sec}),$$ we have $A_T = 245$ for $\Delta R = 1$ cm or $A_T = 2450$ for $\Delta R = 10$ cm. If N pulses are to be accommodated at full energy, the maximum area which one of the pulses must occupy at injection is

$$A = \frac{A_T}{N}$$

For $N = 10^3$ and $\Delta R = 1$ cm, the area per pulse is 0·245.

5.4 *Sudden switch-on*

In this section we examine a system in which no advantage is taken of adiabatic switch-on. This is done merely to emphasize the necessity of adiabatic switching.

We shall also assume sudden switch-off at the first stacking level, in which case the particles will spread out over 2π in phase and have an energy spread ΔE_B, the energy width of the bucket at the first stacking level. The phase area occupied by the particles (initially mono-energetic) will be

$$A = \frac{2\pi\Delta E_B}{\omega}$$

where ω is angular frequency at the first stacking level.

Now the energy spread ΔE_B is given by the expression

$$\Delta E_B = 2\left(\frac{2eV \,.\, E_s}{h\, K\pi}\right)^{1/2} \varepsilon(\gamma)$$

The quantities $\gamma = \sin \phi_s$ and $\varepsilon(\gamma)$ are plotted in Fig. 1. Taking $eV = 30$ MeV and the first stacking level at 100 MeV, which would allow 100 pulses per second, we may calculate this area. We have for the energy spread,

$$\Delta E_B = 2\left[\frac{(2 \times 0{\cdot}03 \times 1038)}{(10 \times 4{\cdot}38 \times 3{\cdot}142)}\right]^{1/2} \varepsilon(\gamma)$$
$$= 1{\cdot}345\varepsilon(\gamma) \quad \text{MeV}$$

giving for the area occupied,

$$\text{Area} = 4{\cdot}75\varepsilon(\gamma).$$

Thus, if we wish to stack 10^3 buckets/cm at full energy, we must have

$$\varepsilon(\gamma) \sim 1/20.$$

From Fig. 1, we see that this would give a value $\gamma = 0{\cdot}98$ and with this value, only 10 per cent of the particles would be trapped (see again Fig. 1). Since similar loss factors are likely to be incurred in subsequent stages, the total loss will be several factors of 10.

5.5 *Adiabatic switch-on*

We shall now examine adiabatic switch-on, again assuming that the initial particles are mono-energetic. A full treatment would require numerical solutions of the non-linear phase equation (2). For the present purpose we shall linearize this equation to give

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega}\right) = \frac{d}{dt}\left(\frac{E_s}{hK\omega_s{}^2}\dot{\Phi}\right) = -\frac{eV}{2\pi}|\cos \phi_s|\,\Phi \quad (12)$$

where Φ is the phase of the particle relative to the synchronous phase ϕ_s and h is the harmonic number.

At injection, the only parameter which is appreciably dependent on time is $V = V(t)$ (strictly $V \cos \phi_s$. If the switch-on procedure were to be computed numerically, independent variation of V and γ would be considered). Putting

$$\Omega^2(t) = \frac{hK\omega_s{}^2 eV}{2\pi E_s} \cos \phi_s, \quad (13)$$

we obtain

$$\ddot{\Phi} + \Omega^2(t)\Phi = 0. \quad (14)$$

At time $t = 0$, let us suppose that Ω is switched on at some value Ω_0. In this case, the particles move round ellipses in phase space with a frequency Ω_0. The initial line distribution will rotate as a straight line if the dynamics is truly linear, sweeping out an area in phase space. Because of the non-linearity of the phase equation, the line will eventually spread over the whole area, mixing with empty phase space. It is this effective area, as Ω is varied, that we have to estimate.

Suppose now that Ω is made to increase with time from its initial value Ω_0. Provided the increase is adiabatic the phase oscillations will decrease in amplitude, but the energy oscillation $[\dot{\Phi}]$ will increase such that the phase area remains constant. The solution will then be represented by the W.K.B. approximation

$$\left.\begin{aligned} \Phi &\sim \frac{1}{\Omega^{1/2}} \exp i \int \Omega\, dt \\ \dot{\Phi} &\sim i\,\Omega^{1/2} \exp i \int \Omega\, dt \end{aligned}\right\} \quad (15)$$

The condition for the change to be adiabatic and conform to the W.K.B. approximation is that the fractional change of frequency per period is small

$$\frac{1}{\Omega}\frac{d\Omega}{dt} \ll \Omega.$$

If we require this condition to be satisfied uniformly, we obtain

$$\frac{1}{\Omega^2}\frac{d\Omega}{dt} = \varepsilon$$

or on integration

$$\Omega^2 = \frac{\alpha}{(T-t)^2} \quad \text{where } \alpha \gg 1.$$

For this particular variation of Ω with time, we are fortunate in having an exact analytical solution (see Appendix 1) with which to test the accuracy of the adiabatic approximation. We find that for $\alpha = 25$, the area only increases by 10 per cent from its value A_0 at $\Omega = \Omega_0$, as Ω is raised to its final value. A graph of $\Omega(t)$ is shown in Fig. 5. To preserve a small phase area, we shall only be interested in large changes in Ω and we see that the curve rises very steeply towards its asymptote. We therefore equate the time to reach the final value of Ω with the time T.

Putting in values of the constants appropriate to 40 MeV, we plot in Fig. 6, V_0 (which gives Ω_0) against T and also the associated phase area A_0. This area A_0 must be greater than the initial area occupied by the injected pulse,

$$\frac{2\pi\Delta E_{\text{inj}}}{\omega}$$

where ΔE_{inj} is the energy spread from the injector. The ΔE_{inj} given by equating these two areas is also

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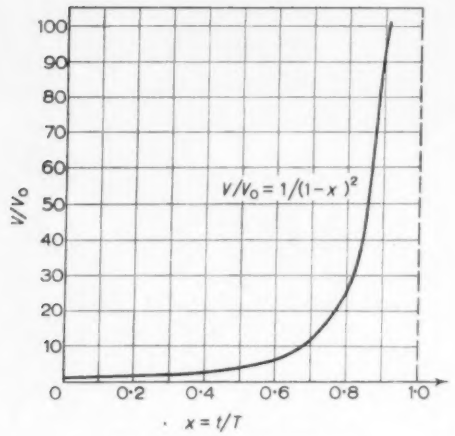

FIG. 5.—Adiabatic voltage rise.

shown in brackets in Fig. 6. The area of the accelerating bucket in the first stage must, of course, be greater than A_0.

Several further questions must be answered to obtain a complete picture of adiabatic switching. Firstly, do other modes of voltage rise allow shorter switching times for the same final area? Examination of the cases of linear and other simple modes (see Appendix 1) suggests that this is not so. Secondly, the effect of non-linearities and the finite energy spread at injection on the fraction of particles trapped must be calculated. This should be done using a digital computer.

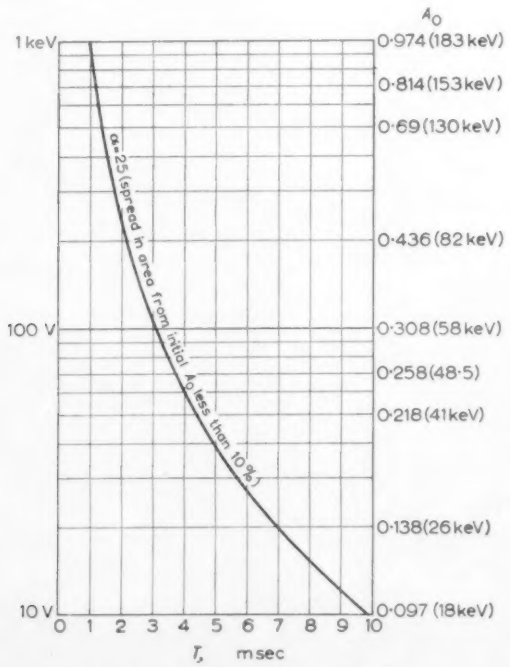

FIG. 6.—Time taken for adiabatic switch-on.

5.6 *Limitations imposed by the first stage*

In Section 2, it was shown that the total circulating current is obtained from the product of three factors

$$n_p \times n_s \times T_c.$$

An upper limit to n_p is 10^{12} (see Sections 5.1 and 5.2), an upper limit of $n_p \times n_s$ is 10^{14} (see Section 2) and an upper limit to T_c is 1000 sec. Thus the total current will not exceed 10^{17} particles. We shall now use the analysis of the preceding sections on the first stage to impose further restrictions. These restrictions are shown in Fig. 7.

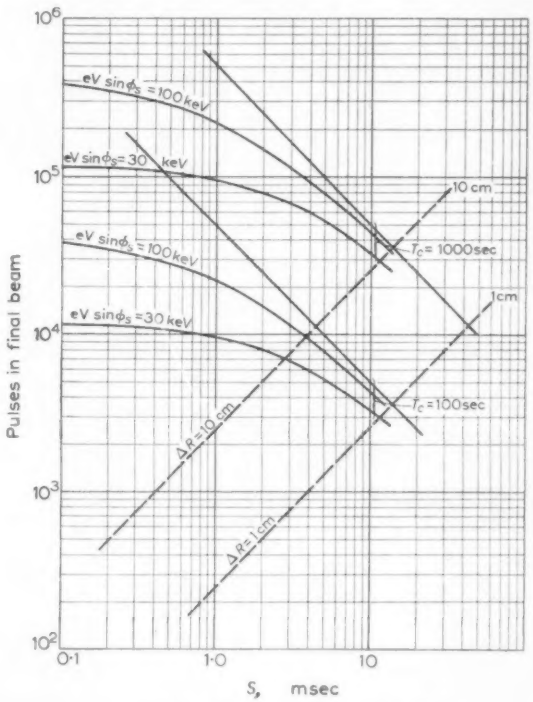

FIG. 7.—Limitations in the first stage.

In Fig. 7, $n_s \times T_c$, the number of pulses making up the circulating beam is plotted against the time allowed for switch-on in the first stage. We showed in Section 5.5 that a given time of switch-on gave rise to a certain phase area A_0 for the pulse. Assuming that this area occupied by the pulse remains constant during the whole acceleration process, we may calculate (see Section 5.3) the number of pulses that may be accommodated within a given radial width at full energy. This gives one set of curves for Fig. 7. Again for a given switch-on time and assuming a time for acceleration we may calculate the pulse repetition rate (n_s). We assume for curves (1) an infinite acceleration voltage (i.e. no time allowed for acceleration), and for curves (2) $eV.\sin\phi_s = 100$ keV and for curves (3) 30 keV in

a first stage from 40 MeV to 100 MeV. The 100 MeV is the lowest stacking energy allowed, if we make the restriction that each stage should occupy at least one half-metre aperture width (see Table 1). Multiplying these values of n_s by the circulating times $T_c = 100$ and 1000 sec, we produce the further two sets of curves shown in Fig. 7.

Let us examine now a system with a 4 msec switching time in the first stage. The pulses can be packed to a density 10^3 per cm radial spread at full energy (see Fig. 7). With a circulating time $T_c = 100$ sec, a beam of 6×10^3 pulses could be obtained with $eV \sin \phi_s = 30$ keV in the first stage. With $eV \sin \phi_s = 100$ keV or $T_c = 200$ sec this could be increased to 10^4 pulses. If T_c is as long as a 1000 sec, it would be possible to obtain a circulating beam of nearly 10^5 pulses; the radial spread however would be much too large, nearly 100 cm. The 10^3 sec circulation could be used to obtain 3×10^4 pulses with $R = 10$ cm by increasing the switching time to 10 msec, but only a factor 3 is gained and the tolerances on energy spread at injection are now very severe. If a circulating time of 1000 sec is available, it should be used to decrease the number of particles accelerated per second. Thus the circulating beam of 10^{16} would be made up (see Section 2) as

$$10^{12} \times 10 \times 10^3.$$

The limitations imposed by the first stage may be summarized as follows.

(1) 10^5 pulses cannot be obtained within a 10 cm spread.

(2) 10^4 pulses can only be obtained within a 1 cm radius spread if $T_c = 1000$ sec and the energy spread at injection is <5 keV.

(3) 10^4 pulses within 10 cm radial spread is possible say with $n_s = 10^2$ pulses per sec and $T_c = 100$ sec or $n_s = 10$ pulses per sec and $T_c = 1000$ sec.

We shall examine now the later stages of acceleration to 15 BeV, taking a switching time of 4 msec in the first stage and $A_0 = 0{\cdot}25$ as the associated phase area per pulse.

6. THE LATER STAGES

With injection at 40 MeV and the first stacking level at 100 MeV, the acceleration time is given by

$$T = \frac{2\pi}{eV \,.\, \gamma} \int_{40\,\mathrm{MeV}}^{100\,\mathrm{MeV}} \frac{dE}{\omega}$$

$$= \frac{8{\cdot}45}{\gamma} \text{ msec with } eV = 30 \text{ keV}.$$

The stable phase area of the bucket A_B is given by

$$A_B = \frac{8}{\omega}\left[\frac{2eV \,.\, E_s}{\pi h K}\right]^{1/2} \alpha(\gamma)$$

With $h = 10$, $eV = 30$ keV, we obtain

$$A_B = 2{\cdot}75\ \alpha(\gamma) > A_0 = 0{\cdot}25$$

With $\gamma = 0{\cdot}8$, $\alpha = 0{\cdot}1$ this condition is satisfied. The time for one cycle $= \dfrac{8{\cdot}45}{0{\cdot}8} + 8$ msec. The repetition rate in the first stage is then 54 pulses per sec.

Next we consider acceleration in the range from about 2 GeV to the final energy 15 GeV. In this part of the acceleration, the particle rotation frequency changes less than 5 per cent. Accordingly, the amount of frequency modulation required is small and it should be possible to put the cavity voltage $V_{\max}$ up to 300 keV or more.

The stable phase area of the bucket at 2 BeV ($h = 10$, $K = 0{\cdot}098$) A_B is

$$A_B = 91{\cdot}07\ (eV^{\mathrm{MeV}})^{1/2}\ \alpha(\gamma).$$

The time, T, for acceleration from 2 GeV to 15 GeV is

$$T = \frac{2\pi}{eV \,.\, \gamma} \int_{2\,\mathrm{GeV}}^{15\,\mathrm{GeV}} \frac{dE}{\omega} = \frac{20{\cdot}68}{(eV^{\mathrm{MeV}})\gamma} \text{ msec}.$$

TABLE 1.—SPIRAL RIDGE MACHINE PARAMETERS

Kinetic energy T (MeV) *(GeV)	Angular frequency $= \dfrac{\beta c}{r_0}$ (rad/sec)	Radial position in a 4 m aperture (m)	$K = \dfrac{E}{\omega}\dfrac{d\omega}{dE}$	$\displaystyle\int_{40\mathrm{MeV}}^{E} \frac{de}{\omega(E)}$ (keV msec)
40	$1{\cdot}182 \times 10^6$	$-4{\cdot}0$	11·317	0
100	$1{\cdot}777 \times 10^6$	$-3{\cdot}53$	4·3811	40·5
500	$3{\cdot}080 \times 10^6$	$-2{\cdot}65$	0·7177	197·2
1*	$3{\cdot}566 \times 10^6$	$-2{\cdot}21$	0·2887	345·4
2*	$3{\cdot}837 \times 10^6$	$-1{\cdot}73$	0·0987	613·4
5*	$3{\cdot}958 \times 10^6$	$-1{\cdot}0$	0·0120	1378·0
15*	$3{\cdot}948 \times 10^6$	0	$-0{\cdot}00984$	3904·0

$k = 74{\cdot}34$. $B_{\max} = 14$ kilogauss. $R_{\max}$ (for 15 GeV) $= 75{\cdot}8$ m.

TABLE 2

	$V_{max} = 30$ keV	100 keV	300 keV	1 MeV
Stable phase area	6·78	12·38	21·46	39·16
Acceleration time, T (sec)	1·72	0·517	0·172	0·0517
Pulses/sec	~15	~94	~490	~3000

When the time occupied by adiabatic switching is negligible, the maximum transfer of phase area between two energies is obtained by taking $\gamma = 0{\cdot}4$ (SYMON and SESSLER, 1956). With this value of γ, we give in Table 2 the rate at which pulses ($A_0 = 0{\cdot}25$) can be accelerated from 2 GeV–15 GeV with different cavity voltages.

There should thus be plenty of time in hand for this stage of the acceleration provided $V_{max} > 100$ keV.

Next, the stages necessary to accelerate the required number of pulses per second from 100 MeV to 2 GeV must be determined. In these stages, we again take $\gamma = 0{\cdot}4$ and $eV = 30$ keV. The stable phase area at 100 MeV, A_B is

$$A_B = \frac{8}{1{\cdot}78}\left[\frac{2 \times 1038 \times 0{\cdot}03}{10 \times \pi \times 4{\cdot}38}\right]^{1/2} \times 0{\cdot}43 = 1{\cdot}52$$

Five pulses may thus be stacked at 100 MeV and accelerated together in the next stage. The time available for one r.f. sweep in the next stage is thus

$$= \frac{5}{54} \text{ sec}$$

$$= 92{\cdot}6 \text{ msec.}$$

Allowing 10 msec for switching, we have an accelerating time

$$T = \frac{2\pi}{eV \cdot \gamma}\int_{100\,\text{MeV}}^{E} \frac{dE}{\omega} = 82{\cdot}6$$

$$\therefore \qquad \int_{100\,\text{MeV}}^{E} \frac{dE}{\omega} = 157{\cdot}7.$$

From Table 1, this gives the next stacking level at 500 MeV. One pulse only (i.e. 5 initial pulses) would be stacked at this level. The next stage would be from 500 MeV–1 GeV at which two pulses (i.e. 10 initial pulses) would be stacked. A further stage from 1 Gev–2 GeV would complete the system.

The complete system would thus be:—

Stage 1 40 MeV — 100 MeV 54 pulses per sec
$A_0 = 0{\cdot}25$ $eV = 30$ keV

Stage 2 100 MeV — 500 MeV 11 pulses per sec taking 5 pulses at a time $eV = 30$ keV

Stage 3 500 MeV — 1GeV 11 pulses per sec taking 5 pulses at a time $eV = 30$ keV

Stage 4 1 GeV — 2 Gev $5\frac{1}{2}$ pulses per sec taking 10 pulses at a time $eV = 30$ keV

Stage 5 2 Gev — 15 GeV $eV > 100$ keV.

If the voltage eV_{max} in stages 1–4 could be raised to 100 keV, the same system would accelerate the same pulses at a repetition rate of 90 pulses per sec. This system is somewhat complicated having five stages and losses might be rather high. If the circulating time is 1000 sec and a pulse repetition rate of only 10 per sec is needed, the number of stages may be reduced to three. The system would be:—

Stage 1 40 MeV — 800 MeV 10 pulses per sec
$A_0 = 0{\cdot}25$ $eV = 30$ keV

Stage 2 800 MeV — 2 GeV

Stage 3 2 GeV — 15 GeV.

We have seen how the phase area of the pulse gives the final beam an energy spread and thus a radial spread. In order to keep this phase area as small as possible, adiabatic switching must be used in the first stage. This in turn reduces the pulse repetition rate, and thus the mean current accelerated.

7. CONCLUSION

Throughout this paper, the final current has been thought of as the product of three factors, the number of particles injected per pulse, the number of pulses accelerated per second and the circulating time at the top energy. The first factor is limited by space charge effects and the current available from an injector. The discussion therefore has been concerned chiefly with the product of the last two factors, the number of pulses making up the final beam. This is constrained by the need to avoid an excessive final energy spread. It has been shown that to keep this energy spread small a strict process of adiabatic switching must be used

during the first stage. The time taken for this adiabatic switching in turn limits the number of pulses per second that can be accelerated by the first stage. Taking into account all these factors, the total number of particles comprising the final beam could be in the region of 10^{16}. This would be made up to 10^{12} particles per pulse and 10^4 pulses, and would be contained in a radial spread of 10 cm. This total of 10^{16} would be sufficient for a colliding beam accelerator but it must be emphasized that very severe practical requirements have to be met. To obtain a fully realistic picture, account will have to be taken of the losses of particles, which will inevitably be sustained. These may well be more severe than those encountered in present accelerators in view of the complication of the radio-frequency system. Further detailed investigations of the various problems which have been discussed here, switch-on and phase-displacement etc., should be made using digital computation. These, together with the many practical requirements, will have to be taken into account in any final verdict on such an accelerator.

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APPENDIX 1

Adiabatic switching

The equation for adiabatic voltage rise discussed in Section 5.5 is

$$\ddot{\Phi} + \Omega_0^2 \left(\frac{T}{T-t}\right)^2 \Phi = 0. \tag{17}$$

This equation has the general solution

$$\Phi = C\left(\frac{T-t}{T}\right)^{1/2} \cos\left[\sqrt{\Omega_0^2 T^2 - 1/4}. \times \log\left(\frac{T}{T-t}\right) + \delta\right]. \tag{18}$$

For the initial conditions $\Phi(0) = \Phi_0$, $\dot{\Phi}(0) = 0$, this becomes

$$\Phi = \frac{\Phi_0}{\cos\delta_0}\left(\frac{T-t}{T}\right)^{1/2} \times \cos\left[\sqrt{\Omega_0^2 T^2 - 1/4}\,.\,\log\left(\frac{T}{T-t}\right) + \delta_0\right]$$

where,

$$\tan\delta_0 = \frac{-1}{2\sqrt{\Omega_0^2 T^2 - 1/4}}.$$

Putting

$$\Delta = \sqrt{\Omega_0^2 T^2 - 1/4}\,\log\left(\frac{T}{T-t}\right) + \delta_0$$

we have

$$\dot{} = \frac{\Phi_0}{\cos\delta_0}\left(\frac{T-t}{T}\right)^{1/2}\left[\frac{-\sqrt{\Omega_0^2 T^2 - 1/4}}{(T-t)}\sin\Delta - \frac{1}{2(T-t)}\cos\Delta\right] \tag{19}$$

The instantaneous phase area A is then

$$A = \frac{\pi(\dot{\Phi} + \Omega^2\Phi^2)}{\Omega}$$

$$= \frac{\pi\Phi_0^2\Omega_0}{\cos^2\delta_0}\left\{\cos^2\Delta + \left[\frac{\sqrt{\Omega_0^2 T_0^2 - 1/4}}{\Omega_0 T}\sin\Delta + \frac{1}{2\Omega_0^2 T^2}\cos\Delta\right]^2\right\}$$

$$= \frac{\pi\Phi_0^2\Omega_0}{\cos^2\delta}\left\{1 + \frac{1}{4\Omega_0^2 T^2}\cos 2\Delta + \frac{\sqrt{\Omega_0^2 T^2 - 1/4}}{2\,\Omega_0^2 T^2}\sin 2\Delta\right\}.$$

Finally

$$A = \frac{\pi\Phi_0^2\Omega_0}{\cos^2\delta_0}\left\{1 + \frac{1}{2\Omega_0 T}\cos\Delta'\right\} \tag{20}$$

where $\Delta' = 2\Delta + x$ with x given by $\tan x = -2\sqrt{\Omega_0^2 T^2 - 1/4}$ and hence $x = \pi/2 - \delta$.

Thus

$$\Delta' = 2\sqrt{\Omega_0^2 T^2 - 1/4}\;.\;\log\left(\frac{T}{T-t}\right) + \delta_0 + \frac{\pi}{2}.$$

Now $\sec^2\delta_0 = 1 + \tan^2\delta_0 = \dfrac{4\Omega_0^2 T^2}{4\Omega_0^2 T^2 - 1}$ so we finally obtain

$$A = \pi\Phi_0^2\Omega_0\left\{\frac{4\Omega_0^2 T^2}{4\Omega_0^2 T^2 - 1}\right\}\left\{1 + \frac{1}{2\Omega_0 T}\cos\Delta'\right\} \tag{21}$$

The only dependence of A upon the time t is through the term in $\cos\Delta'$. Thus the area A oscillates about a mean value

$$A = \pi\Phi_0^2\Omega_0 \times \frac{4\Omega_0^2 T^2}{4\Omega_0^2 T^2 - 1}. \tag{22}$$

If we require that this oscillation should not change A by more than 10 per cent, then $2\Omega_0 T = 10$,

$$\Omega_0 T = 5.$$

In this case, the factor $\frac{4\Omega_0^2T^2}{4\Omega_0^2T^2-1}$ is very nearly equal to one and we may take

$$A = \pi\Phi_0^2\Omega_0.$$

To transform this area into canonical phase area, we use

$$\frac{\Delta E}{\omega} = \frac{E_s}{hK\omega_s^2}\dot{\Phi}$$

and obtain

$$A_0 = \frac{E_s}{hK\omega_s}\pi\Phi_0^2\Omega_0$$

where

$$\Omega_0^2 = \frac{hK\omega_s^2\cos\phi_s}{2\pi E_s}eV_0.$$

In Fig. 6, V_0 and A_0 are plotted against T, for $\Omega_0 T = 5$ that is $\alpha = 25$ (see Section 5.5).

In the case of a linear voltage rise the change of area A with time can again be readily calculated. At first the area grows rapidly, but after the particle has made approximately one half of a phase oscillation the area stays very nearly constant. A certain V_A can be defined such that the phase area does not change by more than δ per cent after V has reached V_A. This V_A is dependent on the gradient dV/dt. We have

$$\frac{d^2\Phi}{dt^2} + c\left(\frac{dV}{dt}\right)t\,\Phi = 0 \tag{23}$$

where c is the appropriate constant. Writing

$$x = \left(c\frac{dV}{dt}\right)^{1/3}t \tag{24}$$

equation (23) reduces to the standard Airy equation

$$\frac{d^2\Phi}{dx^2} + x\Phi = 0 \tag{25}$$

After a certain interval x (which depends only on δ) the solution satisfies the adiabatic criterion. Thus

$$t_A = x_\delta c^{-1/3}\left(\frac{dV}{dt}\right)^{-1/3}$$

and

$$V_A = \frac{dV}{dt}t_A$$

giving

$$V_A = x_\delta c^{-1/3}\left(\frac{dV}{dt}\right)^{2/3} \tag{26}$$

If dV/dt now is allowed to vary, equation (26) gives on integration the adiabatic form $\alpha/(T-t)^2$, previously introduced. We now see that this form has the property of being just adiabatic all the time. It therefore appears that the $\alpha/(T-t)^2$ mode yields the most rapid rise consistent with the adiabatic requirement.

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ESSENTIAL LIMITATIONS AND *T-Nt* DIAGRAM FOR SELF-SUSTAINING FUSION SYSTEMS*

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Abstract—The object of this paper is to show various criteria and essential limitations for any fusion reactor in a unified way. For this purpose, any fusion system is described in terms of two fundamental variables, a new parameter $\tau = Nt$, called the reduced time, and the temperature T. Differential equations for time variations of the compositions of fuels, as well as the bremsstrahlung loss caused by electron-electron scattering, are newly introduced into the energy balancing considerations. As a result, we can find maximum and minimum values of T and τ, which are expressed as two contours enclosing energy balancing regions in narrow and wide senses in T–τ space, called the T–Nt diagram. These contours include the criteria of POST and LAWSON. With the aid of the T–Nt diagram we discuss the ideal temperature to operate reactors, upper limits to efficiencies for available energy gains, two roles played by ashes, the critical time to renew the fuel, the fact that, without effective utilization of neutrons and their energy, pure deuterium fuel cannot be used as a direct energy source, time variations of energy generation and losses, and some other criteria of a technical character. The T–Nt diagram can be regarded as a strategic chart for fusion projects.

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1. INTRODUCTION

IN recent articles, some of the conditions necessary for realizing a power producing thermonuclear reactor have been studied. Lower limits to the plasma temperature necessary to maintain a power-balancing reaction have been discussed by POST (1956). He defined them as the temperature at which the nuclear power density from fusion reactions is equal to the radiative power density loss of bremsstrahlung caused by ion-electron scattering. These lower limits are about $3{\cdot}5 \times 10^{8}$°K for a pure deuterium plasma and 5×10^{7}°K for a plasma consisting of equal parts of tritium and deuterium. More recently, in the article by LAWSON (1957) it is shown that not only must the temperature be above a certain minimum value, but also the reaction must be maintained long enough for a definite fraction of fuel to be burnt, in order to regain more energy than that injected and consumed in the process of the initial heating.

In this article, we adopt a new method in order to obtain all criteria fundamental for any self-sustaining fusion reactor in a unified way. For this purpose, a pair of fundamental variables sufficient to describe the essential characteristics of any energy-balancing fusion system is introduced: a new parameter called 'reduced time', $\tau = Nt$, and the plasma temperature, T. Here N is the number density of ions and t is time. As a result, we can find several fundamental criteria, including those of POST and LAWSON, and distinguish between concepts of energy balancing in two different senses. These can be described in such a way that they form closed contours surrounding energy balancing regions on T–τ space.

In Section 2 equations for a self-sustaining nuclear fusion system are derived. Section 3 is devoted to the discussions on the fundamental variables of thermonuclear systems. The application of the T–Nt diagram to thermonuclear reactors is discussed in Sections 4 and 5. In Section 6 the utility of pure deuterium fuel and in Section 7 double roles of secondary nuclei are discussed. In Section 8, as a concluding remark, the T–Nt diagram for reactors is compared with those for thermonuclear bombs and energy generating stars.

2. BASIC EQUATIONS FOR SELF-SUSTAINING NUCLEAR FUSION SYSTEMS

In obtaining the most general conditions necessary for any self-sustaining nuclear fusion system, we must not impose any conditions other than the equations of energy balancing upon the system, otherwise the generality of the conclusions will be lost. But we must notice that, within such a general definition as above, not only all types of nuclear fusion reactors, but also energy generating stars and thermonuclear bombs are included.

(i) *Definitions of energy balancing in narrow and wide senses*

We use two kinds of definition of energy balancing. The one is named 'energy balancing in a narrow sense', or power balancing, which means the balancing of the power produced with losses at each instant. The other is 'energy balancing in a wide sense', which means the balancing of integrated energy gains with

* Based on a thesis for the degree of Doctor of Science presented to Kyoto University.

integrated losses for the whole operating cycle. The former is a physical definition, useful in deciding whether the temperature and composition are suitable for sustaining the reaction against radiation losses, while the latter is rather an economic definition.

As a general type of system, we consider a combined system consisting of confined plasma with fusion reactions, apparatus surrounding the plasma, and an energy pool. In the plasma, energy is created with a power density $\varepsilon = P_c + P_n$, where P_c and P_n are the portions carried by charged products and by neutrons. Since these reaction products have higher energies than those of the plasma some of them cannot be trapped by magnetic field and collisions, and so transfer the energy released out of the plasma. At the same time previously stored energy is also transferred out of the plasma by various radiations, thermal diffusion of stored ions and electrons, and inductive generation of electric current through expansion in magnetic fields.

Some fusion systems, especially energy generating stars and thermonuclear bombs, store the released energy for a considerable time within plasma itself owing to their high densities. For these systems efficiencies for the transfer of energy are defined within the plasma only. But for fusion reactors on earth, a considerable fraction of the particles cannot be contained within the plasma. Thus, for such reactors, efficiencies η should be defined for whole processes of energy transfer through the apparatus surrounding the plasma and the energy pool. Through these processes, the loss density per unit time is written as

$$(1-\eta_c)P_c + (1-\eta_n)P_n + (1-\eta_d)P_d + (1-\eta_b)(P_{bie}+P_{bee}) + (1-\eta_r)P_r + (1-\eta_s)P_s + (1-\eta_e)\frac{dW_e}{dt}$$

where P's are power densities for various kinds of transfer mechanisms and W_e is the energy density carried out of the plasma into the pool in the form of electric current. The subscript c refers to charged reaction products, n to neutrons, *bie* to bremsstrahlung caused by ion-electron scattering, *bee* to bremsstrahlung caused by electron-electron scattering, s to synchrotron radiation, d to thermal diffusion of electrons and ions stored within the plasmas, and e to the transfer process by generation by induction of electric current.

We must consider also $(1-\eta_t)\dfrac{dW_t}{dt}$, the loss in the process of injecting energy into the plasma from the pool. Here W_t and η_t refer to the injected energy per unit volume and its efficiency, respectively. Besides the heat injection, W_t includes the energy supplied from applied electric magnetic or gravitational fields. $\dfrac{dW_e}{dt}$ and $\dfrac{dW_t}{dt}$ have only non-negative values.

Deducting the sum of above loss densities from the power density created in the reacting plasma, the equation of energy balancing in the narrow sense becomes

$$\varepsilon - (1-\eta_c)P_c - (1-\eta_n)P_n - (1-\eta_r)P_r - (1-\eta_s)P_s - (1-\eta_d)P_d - (1-\eta_e)\frac{dW_e}{dt} - (1-\eta_b)(P_{bie}+P_{bee}) - (1-\eta_t)\frac{dW_t}{dt} \geq 0\,.$$

Substituting $\varepsilon = P_c + P_n$, and $P_{bie} + P_{bee} = P_b$

$$\eta_c P_c + \eta_n P_n - (1-\eta_b)P_b - (1-\eta_r)P_r - (1-\eta_s)P_s - (1-\eta_d)P_d - (1-\eta_e)\frac{dW_e}{dt} - (1-\eta_t)\frac{dW_t}{dt} \geq 0 \quad (1)$$

Similarly, for energy balancing in wide sense,

$$\int\{\eta_c P_c + \eta_n P_n - (1-\eta_b)P_b - (1-\eta_r)P_r - (1-\eta_s)P_s - (1-\eta_d)P_d\}\,dt - (1-\eta_e)W_e - (1-\eta_t)W_t \geq 0. \quad (2)$$

P_c, P_n, P_{bie} and P_r in (1) and (2) depend not only on the particle energy but also on the composition of the gas. The composition of the gas and the abundances of each nuclei, are functions of time, given by a set of differential equations. We must notice, however, that in nuclear fusion systems there are two or more groups of the same kind of nuclei. These groups differ from each other in their kinetic energy and also in their reaction cross-sections. The energetic group consists of daughter particles which have not collided sufficiently often that they have lost their initial energy from the reaction. These groups must satisfy different equations as

$$\frac{dN_i}{dt} = \frac{1}{t'}N_i - \sum_j \langle\sigma v\rangle_{ij}N_iN_j - \sum_{j'}\langle\sigma v\rangle_{ij}N_iN_j, \quad (3a)$$

and

$$\frac{dN_{i'}}{dt} = \sum_{j,k}\langle\sigma v\rangle_{jk}N_jN_k + \sum_{j,k\neq i'}\langle\sigma v\rangle_{jk'}N_jN_{k'} + \sum_{j',k\neq i'}\langle\sigma v\rangle_{j'k'}N_{j'}N_{k'} - \sum_j\langle\sigma v\rangle_{i'j}N_{i'}N_j - \sum_{j'}\langle\sigma v\rangle_{i'j'}N_{i'}N_{j'} - \frac{1}{t_{i'}}N_{i'} \quad (3b)$$

TABLE 1.—POWER DENSITIES OF VARIOUS RADIATIONS

Radiations	Power densities (W/cm³)
Bremsstrahlung (ion-electron scattering)	$P_{bie} = 5{\cdot}4 \times 10^{-31} \sum_i Z^2 N_i N_e T_e^{1/2}$
Bremsstrahlung (electron scattering)	$P_{bee} = 7{\cdot}6 \times 10^{-33} N_e^2 T_e^{3/2}$; THOMPSON (1957b)
Photons by recombinations (in hydrogen)	$P_r = 0{\cdot}5 \times 10^{-31} N_e N_i T_e^{-1/2}$, $E_0 \gg T_0$; $P_r = 1{\cdot}8 \times 10^{-32} N_e N_i T_e^{-1}$, $E_0 \ll T$
Synchrotron radiation	$P_c = 3{\cdot}2 \times 10^{-35} N_e T_e H^2$

T is electron temperature in keV, H in gauss and Z is the atomic number of an ion. A Maxwellian distribution of ions and electrons is assumed.

Here, N_i, N_j and N_k are number densities of ions with averaged energy, $N_{i'}$, $N_{j'}$ and $N_{k'}$ are those of ions with deviated energy, and i, j or k indicates each kind of nuclei i.e., deuteron, triton, etc. $\langle\sigma v\rangle$ is a mean of σv averaged over a given velocity distribution in the centre of mass system, where σ is a reaction cross-section and v is a velocity in the C.M.S. Hence $P_{cij} = (Q - Q_n)_{ij}\langle\sigma v\rangle_{ij} N_i N_j$ and $P_{nij} = Q_{nij}\langle\sigma v\rangle_{ij} N_i N_j$ are power densities carried by charged particles and neutrons, respectively, through the reaction rate between i and j nuclei. Q and Q_n are Q values per reaction and its shared energy of a neutron, respectively. t' is the relaxation time of ions with deviated energy.

(ii) *Approximations common to any energy balancing fusion system*

(a) Cross-sections for reactions are much lower than those for scattering. In a highly ionized stable plasma no strong electric field can exist which might accelerate particles against the effect of collisions. A Maxwellian distribution will therefore be assumed with equal electron and ion temperatures. Tables 1 and 2 are devoted to various radiation densities and reaction rates of interest. Here and after T is written in keV and $N = \Sigma_i N_i$ is the total number of ions. In thermal equilibrium the equation of plasma temperature can be defined as

$$\frac{3}{2}\frac{d\{(N + N_e)T\}}{dt} = \eta_c' P_c + \eta_n' P_n + \eta_t \frac{dW_t}{dt} - \frac{dW_e}{dt} - (1 - \eta_t')(P_{bie} + P_{bee}) - (1 - \eta_d')P_d - (1 - \eta_r')P_r - (1 - \eta_s')P_s \quad (4)$$

where η_i''s are defined only within the plasma, and are lower than η_i's. When $dW_e/dt = 0$ and $dT/dt \geqslant 0$ in (1), the temperature is defined as self-sustaining.

(b) Effects of energetic non-averaged daughter nuclei are neglected.

The relaxation time of particles is proportional to $T^{3/2}$, but even at $10^{9°}$K the relaxation times at most are a thousand times as short as mean reaction lives. For this reason, although the number of high energy nuclei increases for a time of the same order as the relaxation times, the ratio of the number of energetically deviated nuclei to those with the averaged energy cannot exceed the ratio of the relaxation time to the mean reaction time for producing the energetic daughter nuclei. So we can neglect effects of the energetic deviation on reaction rates, and the two groups may be summed together in one equation as (7).

(c) The synchrotron radiation is neglected.

This cannot be included without loss of generality. Further it will be below the plasma frequency and mostly cannot escape.

(d) Similarly, diffusion loss of electrons and ions of plasmas is neglected since it depends on the details of the actual apparatus. Therefore we adopt $(1 - \eta_d)P_d = 0$.

Adopting above approximations, we can rewrite (1)–(4) as

$$\eta_c \sum_{ij}(Q - Q_n)_{ij}\langle\sigma v\rangle_{ij} N_i N_j + \eta_n \sum_{ij} Q_{nij}\langle\sigma v\rangle_{ij} N_i N_j - (1 - \eta_b)5{\cdot}4 \times 10^{-31} T^{1/2}\left(N_e \sum_i Z_i^2 N_i + \frac{T}{66} N_e^2\right) - (1 - \eta_r)P_r - (1 - \eta_t)\frac{dW_t}{dt} - (1 - \eta_e)\frac{dW_e}{dt} \geq 0 \quad (5)$$

$$\int\left\{\eta_c \sum_{ij}(Q - Q_n)_{ij}\langle\sigma v\rangle_{ij} N_i N_j + \eta_n \sum_{ij} Q_{nij}\langle\sigma v\rangle_{ij} N_i N_j - (1 - \eta_b)5{\cdot}4 \times 10^{-31} T^{1/2}\left(N_e \sum_i Z_i^2 N_i + \frac{T}{66} N_e^2\right) - (1 - \eta_r)P_r\right\} dt - (1 - \eta_t)W_t - (1 - \eta_e)W_e \geq 0 \quad (6)$$

$$\frac{dN_i}{dt} = \sum_{j,k \neq i}\langle\sigma v\rangle_{jk} N_j N_k - \sum_j \langle\sigma v\rangle_{ij} N_i N_j \quad (7)$$

and

$$\frac{3}{2}\frac{d\{(N+N_e)T\}}{dt} = \eta_c' \sum_{ij}(Q-Q_n)_{ij}\langle\sigma v\rangle_{ij} + \eta_n' \sum_{ij} Q_{nij}\langle\sigma v\rangle_{ij}N_iN_j - (1-\eta_b')5{\cdot}4\times 10^{-31}T^{1/2}\left(N_e\sum_i Z_i^2N_i + \frac{T}{66}N_e^2\right) - (1-\eta_r')P_r + \eta_t\frac{dW_t}{dt} - \frac{dW_e}{dt} \quad (8)$$

This set of formulae is the most general one for self-sustaining fusion systems which are approximately 'thermonuclear'.

3. FUNDAMENTAL VARIABLES FOR A THERMONUCLEAR SYSTEM CONSIDERED AS AN EVOLUTIONARY SYSTEM

In the absence of externally imposed sources or sinks of energy or of particles, the state variables and composition of any thermonuclear system will change with time. The system will evolve in a manner analogous to stellar evolution, being in a quasi-stationary state when considered macroscopically, but undergoing microscopic transformation of an irreversible nature.

Besides the macroscopic variables N and T we should introduce microscopic variables to describe the evolution of the system. Fortunately, we do not need to consider individual particle energies because of the assumptions of thermal equilibrium, and can take simply T, N and t. Now since the probability of all two body processes (reacting collisions, scattering, radiation of bremsstrahlung) is directly proportional to the density, N, any process connected with such interactions depends on the compound variable $\tau = Nt$, but not t only.

Adopting $\tau \equiv Nt$, $n_i = \frac{N_i}{N}$, $n_e = \frac{N_e}{N}$, $w_t = \frac{W_t}{N}$ and $w_e = \frac{W_e}{N}$, equation (5), (6), (7) and (8) are rewritten as

$$\eta_c\sum_{ij}(Q-Q_n)_{ij}\langle\sigma v\rangle_{ij}n_in_j + \eta_n\sum_{ij}Q_{nij}\langle\sigma v\rangle_{ij}n_in_j - (1-\eta_b)5{\cdot}4\times 10^{-31}T^{1/2}\left(n_e\sum_i Z_i^2n_i + \frac{T}{66}n_e^2\right) - (1-\eta_r)P_r/N^2 - \frac{1}{1-\frac{d\log a}{d\log\tau}}\left\{\left[(1-\eta_t)w_t + (1-\eta_e)w_e\right]\frac{d\log a}{d\tau} + (1-\eta_t)\frac{dw_t}{d\tau} + (1-\eta_e)\frac{dw_e}{d\tau}\right\} \geq 0 \quad (9)$$

$$\int\left\{\eta_c\sum_{ij}(Q-Q_n)_{ij}\langle\sigma v\rangle_{ij}n_in_j + \eta_n\sum_{ij}Q_{nij}\langle\sigma v\rangle_{ij}n_in_j - (1-\eta_b)5{\cdot}4\times 10^{-31}T^{1/2}\left(n_e\sum_i Z_i^2n_i + \frac{T}{66}n_e^2\right) - (1-\eta_r)P_r/N^2\right\}\times\left(1-\frac{d\log a}{d\log\tau}\right)d\tau - (1-\eta_t)w_t - (1-\eta_e)w_e \geq 0 \quad (10)$$

$$\frac{dn_i}{d\tau} = \left\{\sum_{j,k\mp i}\langle\sigma v\rangle_{jk}n_jn_k - \sum_j\langle\sigma v\rangle_{ij}n_in_j\right\}\times\left(1-\frac{d\log a}{d\log\tau}\right) - n_i\frac{d\log a}{d\tau} \quad (11)$$

$$\frac{3}{2}\frac{d\{(1+n_e)T\}}{d\tau} = \{\eta_c'\sum_{ij}(Q\ Q_n)_{ij}\langle\sigma v\rangle_{ij}n_in_j + \eta_n'\sum_{ij}Q_{nij}\langle\sigma v\rangle_{ij}n_in_j - (1-\eta_b')5{\cdot}4\times 10^{-31}T^{1/2}\left(n_e\sum_i Z_i^2\,n_i + \frac{T}{66}n_e^2\right) - (1-\eta_r')P_r/N^2\}\left(1-\frac{d\log a}{d\log\tau}\right) + [\eta_tw_t - w_e]\frac{d\log a}{d\tau} + \eta_t\frac{dw_t}{d\tau} - \frac{dw_e}{d\tau} \quad (12)$$

Here $a = N/N_0 = bV_0/V$ where V is the volume of the system; the subscript zero denotes initial conditions, and b is a factor indicating the change of the number of particles due to diffusion. The quantity a describes the change in density of the system due to expansion and diffusion. The reduced time $\tau = Nt$ is therefore a concept of time applicable to statistical systems (whether or not reactions occur), as distinct from time measured by rotation of a clock. The fact that the reduced time depends on $N = aN_0$ reveals that the evolution of the system is due to mutual interactions of the particles.

For actual reactors, w_t, w_e, a are parameters of a technical nature, and their η values depend on the particular type of reactor under consideration. If they are given, then equation (8–10) can be used to calculate the range of T and τ for a self-sustaining system. In particular we can draw a boundary contour in the T–τ plane, within which the system will be energy balancing, as in Figs. 5–7. Furthermore, since T and τ are fundamental variables, other criteria of a technical nature can be plotted on this plane. This T–Nt diagram forms a 'strategic chart' which we must study when making plans for cutting a path to the energy balancing region.

4. T–Nt DIAGRAM FOR FUSION REACTORS ON EARTH

For terrestrial fusion reactors we make the following simplification and assumptions about the values of the η's.

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TABLE 2.—CHARACTERISTICS OF FUSION REACTIONS

Reactions	*Q*-value	Energy carried by a neutron	$\langle\sigma v\rangle_{ij}$	Energy generation rate $\varepsilon - P_n$ (W cm^{-3}) (excluding the power carried by neutrons)	Power density carried by neutrons, P_n
D + D → T + *p*	4·04 MeV		$\alpha \cdot A$	$3{\cdot}48 \times 10^{-13} a A N_D^2$	
→ ^{3}He + *n*	3·27 MeV	2·44 MeV	$(1-\alpha)A$	$0{\cdot}649 \times 10^{-13}(1-a)AN_D^2$	$1{\cdot}95 \times 10^{-13}(1-a)AN_D^2$
D + T → ^{4}He + *n*	17·58 MeV	14·06 MeV	*B*	$5{\cdot}64 \times 10^{-13} B N_D N_T$	$22{\cdot}56 \times 10^{-13} B N_D N_T$
D + ^{3}He → ^{4}He + *p*	18·34 MeV		*C*	$29{\cdot}38 \times 10^{-13} C N_D N_{^3He}$	
T + T → ^{4}He + 2*n*	11·32 MeV	~7·6 MeV	*E*	$3{\cdot}12 \times 10^{-13} E N_T^2$	$\sim 6{\cdot}2 \times 10^{-13} E N_T^2$
T + ^{3}He → ^{4}He + D	14·31 MeV		0·4*D*	$\sim 13{\cdot}8 \times 10^{-13} D N_T N_{^3He}$	$\sim 6{\cdot}0 \times 10^{-13} D N_T N_{^3He}$
→ ^{4}He + *p* + *n*	12·08 MeV	~4·0 MeV	0·6*D*		

Energy Generation Rates: α and $(1-\alpha)$ are branching ratios in *DD* reactions, while 0·4 and 0·6 are the branching ratios in T^{3}He reactions.

Reaction rate for *DD* and *TT* reactions is not $\langle\sigma v\rangle N_D^2$, but $\langle\sigma v\rangle \frac{N_D^2}{2}$. Values of $\langle\sigma v\rangle_{ij}$ are given by THOMPSON (1957a).

(a) At temperatures higher than a few million degrees the plasma is completely ionized, and P_r may be set equal to zero. This is not so in stars, for although the high temperature bremsstrahlung produced within the stars is trapped because of their vast size, lower frequency radiation caused by recombination near the cooler stellar surface can escape.

(b) Since the bremsstrahlung can be trapped only within material walls surrounding the confined plasmas, the energy of the bremsstrahlung is at once converted into thermal energy. Therefore the efficiency with which this energy can be transferred should have a low value such as $0 < \eta_b \lesssim \frac{1}{3}$ whose maximum is the efficiency of the usual steam cycle.

(c) Though neutrons cannot be stopped in the plasma; they would probably be used for fuel cycling of tritium through ^{6}Li$(n,t)^4$He reactions, or to produce heat in other reactions. Here we adopt $\eta_n = 0$, because we wish to discriminate between the role of the energy carried by neutrons from that carried by other particles, considering the technical ambiguity of the situation. The author will consider the case when $\eta_n > 0$ in a future paper.

(d) The efficiency η_c of energy carried by charged products can be high, since most of the charged particles are confined by magnetic fields. We take the upper and lower limiting values 1 and 1/3, for η_c.

(e) The efficiency η_t of energy supplied in the process of the initial heating is that expected for the usual boilers and engines, $\eta_t \lesssim 1/3$. We assume $(1-\eta_c) \approx 0$ since energy is transferred in the form of electric currents.

According to the above approximations, (a)–(e) and Table 2, (9), (10), (11) and (12) are rewritten as:

$$\eta_c \sum_{ij} (Q - Q_n)_{ij}\langle\sigma v\rangle_{ij} n_i n_j + \eta_n \sum_{ij} Q_{nij}\langle\sigma v\rangle_{ij} n_i n_j - (1-\eta_b)5{\cdot}4 \times 10^{-31} T^{1/2}\left(n_e \sum_i Z_i^2 n_i + \frac{T}{66} n_e^2\right) - \frac{1}{1 - \dfrac{d\log a}{d\log \tau}}\left\{[(1-\eta_t)w_b + (1-\eta_e)w_e]\frac{d\log a}{d\tau} + (1-\eta_t)\frac{dw_t}{d\tau} + (1-\eta_e)\frac{dw_e}{d\tau}\right\} \geq 0 \tag{13}$$

$$\int\left\{\eta_c \sum_{ij} (Q - Q_n)_{ij}\langle\sigma v\rangle_{ij} n_i n_j + \eta_n \sum_{ij} Q_{nij}\langle\sigma v\rangle_{ij} n_i n_j - (1-\eta_b)5{\cdot}4 \times 10^{-31} T^{1/2}\left(n_e \sum_i Z_i^2 n_i + \frac{T}{66}\eta_e^2\right)\right\}\left(1 - \frac{d\log a}{d\log \tau}\right)d\tau - (1-\eta_t)w_t - (1-\eta_e)w_e \geq 0 \tag{14}$$

$$\frac{dn_D}{d\tau} = \left(-A\frac{n_D^2}{2} - Bn_D n_T - Cn_D n_{^3He} + 0{\cdot}4Dn_T n_{^3He}\right) \times \left(1 - \frac{d\log a}{d\log \tau}\right) - n_D \frac{d\log a}{d\tau} \tag{15-1}$$

$$\frac{dn_T}{d\tau} = \left(-Bn_D n_T - Dn_T n_{^3He} - E\frac{n_T}{2} + \alpha A\frac{n_D^2}{2}\right) \times \left(1 - \frac{d\log a}{d\log \tau}\right) - n_T \frac{d\log a}{d\tau} \tag{15-2}$$

$$\frac{dn_{^3He}}{d\tau} = \left(-Cn_D n_{^3He} - Dn_T n_{^3He} + (1-\alpha)A\frac{n_D^2}{2}\right) \times \left(1 - \frac{d\log a}{d\log \tau}\right) - n_{^3He} \frac{d\log a}{d\tau} \tag{15-3}$$

$$\frac{dn_\alpha}{d\tau} = \left(Bn_D n_T + Cn_D n_{^3He} + Dn_T n_{^3He} + E\frac{{n_T}^2}{2}\right) \times \left(1 - \frac{d\log a}{d\log \tau}\right) - n_\alpha \frac{d\log a}{d\tau} \quad (15\text{–}4)$$

$$\frac{dn_p}{d\tau} = \left(\alpha A \frac{{n_D}^2}{2} + Cn_D n_{^3He} + 0{\cdot}6Dn_T n_{^3He}\right) \left(1 - \frac{d\log a}{d\log \tau}\right) - n_p \frac{d\log a}{d\tau} \quad (15\text{–}5)$$

$$\text{and } \frac{3}{2}\frac{d\{(1+n_e)\tau\}}{d\tau} = \left\{\eta_c' \sum_{ij}(Q - Q_n)_{ij}\langle\sigma v\rangle_{ij} n_i n_j + \eta_n' \sum_{ij} Q_{nij}\langle\sigma v\rangle_{ij} n_i n_j - (1-\eta_b')5{\cdot}4 \times 10^{-31} T^{1/2}\left(n_e \sum_i Z_i^2 n_i + \frac{T}{66} n_e^2\right)\right\}\left(1 - \frac{d\log a}{d\log \tau}\right) + [\eta_t w_t - w_e]\frac{d\log \tau}{d\tau} + \eta_t \frac{dw_t}{d\tau} - \frac{dw_e}{d\tau} \quad (16)$$

(15) and (16) make a set of combined equations to give T and N_i's as functions of τ, $T = T(\tau, T_0, n_{D0}, n_{T0} \ldots)$ and $N_i = N_i(\tau, T_0, n_{D0}, n_{T0} \ldots)$, where T_0, N_{D0}, $N_{T0} \ldots$ are the initial values of T, N_D, $N_T \ldots$ at $\tau = 0$, respectively. Putting these solutions into (13) we get the energy balancing region in the narrow and wide senses, respectively, which depend on the initial values, T_0, N_{D0}, N_{T0}, etc. T_0 is the atmospheric temperature, and thus the energy balancing regions depend only upon the mixing ratio of the initial fuel, parameters of a technical nature, and the η's. We will use the following method to find the contours which bound the energy balancing regions of practical interest. The time at which the temperature is controlled to change in practical cases is the time when the reaction should be commenced or stopped. Therefore, in the time interval we need to consider $dT/d\tau = 0$ at $\tau > 0$. Next we set $dw_t/dt = dw_e/d\tau = 0$ at $\tau > 0$ because this gives the most generous condition necessary to satisfy the energy balancing in the narrow sense, as easily seen in (13). In order to get the most generous condition necessary to satisfy the energy balancing in the wide sense, for w_t and w_e we adopt the lowest values consistent with the condition $dT/d\tau = dw_t/d\tau = dw_e/d\tau = 0$ at $\tau > 0$; we take $w_t = (1+n_e)\frac{3}{2}T = \text{const.}$ and $(1-\eta_e)w_e = 0$ in (14). We also assume $d\log a = 0$ at $\tau > 0$.

For reactors on earth, the compression by magnetic fields, (the pinch effect), is often used to separate plasmas from the material walls, but the time scale for compression is far smaller than the mean reaction lives of nuclei. Therefore, the assumption $d\log a = 0$ at $\tau > 0$ is not only a good approximation, but also serves to illustrate the main feature of time variations. Also in other thermonuclear systems, except thermonuclear bombs and supernovae, $d\log a = 0$ is a good approximation.

First, in this Section, and Sections 5 and 6, we shall consider the case of all charged products confined within plasmas, and next, in Section 7, the case of no products confined. We adopt two typical efficiencies as (i) $\eta_c = 1$, $\eta_n = 0$, and $\eta_b = \eta_t = 0$ and (ii) $\eta_c = \eta_b = \eta_t = \frac{1}{3}$, $\eta_n = 0$. η_n is set equal to zero for the reasons explained in Section 4(c). We consider two initial mixture ratios of the fuels $N_{D0} = 1$ and $N_{D0} = N_{T0} = \frac{1}{2}$.

At various temperatures, $T = T_1$, $T_2 \ldots$, the equations (15) should be solved to give $n_{i,T=T_1} = f_{i,T=T_1}(\tau), n_{i,T=T_2} = f_{i,T=T_2}(\tau)$ etc. For the case when secondarily produced tritons and ^{3}He's are confined, we ought to integrate (15) numerically for $n_{D0} \gg n_{T0}$ as given in Figs. 1(b) and 2; although when $n_{D0} \simeq n_{T0}$ this has been calculated analytically with an approximation $n_D \simeq n_T$. This difference arises from the fact that $\langle\sigma v\rangle_{DT}$ and $\langle\sigma v\rangle_{D^3He}$ are much larger than $\langle\sigma v\rangle_{DD}$ especially at high temperatures.

Next we put the solutions of (15) into (13). Domains satisfying (11) are shown in Figs. 2 and 3, in which $\frac{\eta_c P_c}{N^2} = \sum_{ij}(Q - Q_n)_{ij}\langle\sigma v\rangle_{ij} n_i n_j$ is plotted against τ. $\tau_{\min}$ and $\tau_{\max}$, where they exist, are the critical reduced times when $\eta_c P_c$ becomes equal to the sum of radiation losses. If these values of $\tau_{\min}$ and $\tau_{\max}$ for each temperature are plotted in T–τ space they define a contour shown as region (II) in Figs. 5 and 6 which borders the energy balancing region in the narrow sense with the maximum temperature, $T_{\max}$, and the minimum temperature $T_{\min}$.

Likewise we can define a contour on the T–τ plane which encloses a region of energy balancing in the wide sense. Maximum and minimum values of T and τ are denoted by $\bar{T}_{\max}$, $\bar{T}_{\min}$, $\bar{\tau}_{\max}$ and $\bar{\tau}_{\min}$.

These limits of temperature $\bar{T}_{\min}$ and $\bar{T}_{\max}$ are also illustrated by the curve of the total available energy $Q_g = \frac{1}{N_0}\int_{\bar{\tau}_{\min}}^{\tau_{\max}}$ (the left-hand side of equation (9))$d\tau$ against temperature in Fig 8 (II).

5. SIGNIFICANCE OF THE ABOVE LIMITATIONS AND OTHER CRITERIA

(a) $T_{\min}$ *and* $\bar{T}_{\max}$

The horizontal tangential line to the contour defining energy balancing in the narrow sense at $\tau = 0$, correspond to POST's criterion of minimum temperature. Different from POST's criterion, however, our $T_{\min}$ is a function of reduced time τ, and

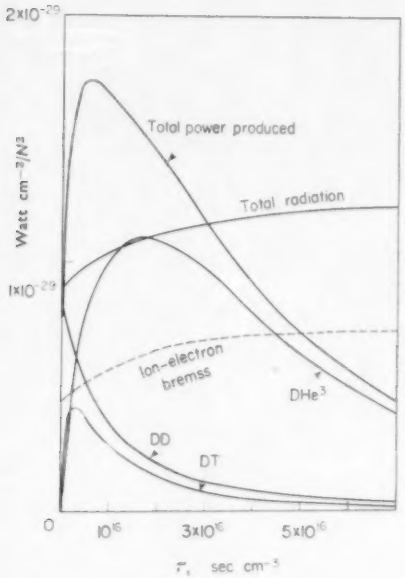

FIG. 1 (a).

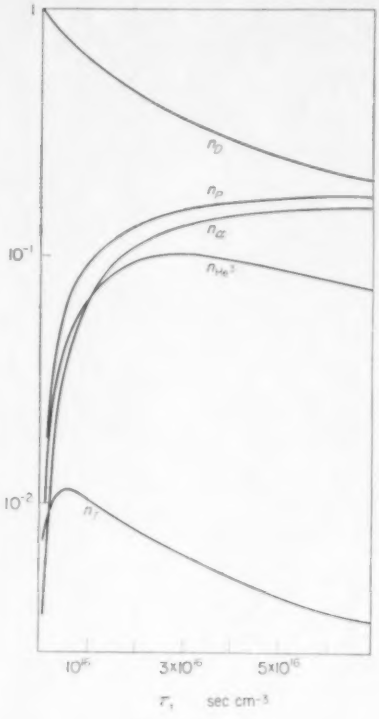

FIG. 1 (b).

FIG. 1.—(a) Variation with reduced time of energy generation (excluding energy carried by neutrons) and radiation losses for a system consisting initially of pure deuterium, $n_{D_0} = 1$. Power densities are expressed in W cm^{-3}/N^2. (b) Variation of composition, $n_{D_0} = 1$. In both cases $T = 8 \times 10^{8}$°K and all charged reaction products are confined. Note that n_T and n^3_{He} increase up to a time of the order of their reaction time, and then decrease, and also that electron-electron bremsstrahlung is as important as ion-electron bremsstrahlung.

includes effect of bremsstrahlung caused by electron-electron scattering. As seen in Figs. 4, 5(a) and 6(a) values of $T_{\min}$ are scarcely higher than POST's value in cases of $\eta_e = 1$ and $\eta_b = \eta_n = 0$ both for $n_{D0} = 1{\cdot}0$ and $n_{D0} = n_{T0} = \frac{1}{2}$, since P_{bee} is rather small at the lower temperature. $\bar{T}_{\min}$ should be somewhat higher than $T_{\min}$, in order to regain the energy consumed in initial heating process, $(1 - \eta_t)w_t = (1 - \eta_t)3T$.

(b) $T_{\max}$ *and* $\bar{T}_{\max}$

Maximum limits to T and $\bar{T}$ occur because above about 10^{9}°K the radiation increases more rapidly with temperature than the power from reactions; this is illustrated in Figs. 4, 5 and 8. When $n_{D0} = n_{T0} = \frac{1}{2}$, the maximum value of $T_{\max}$ occurs at $\tau = 0$; when $n_{D0} \simeq 1$ on the other hand it occurs at $\tau \simeq 10^{16}$, the reason for this is that the energy generation rate increases due to the presence of secondary tritons and ^{3}He. Generally $\bar{T}_{\max}$ is considerably lower than $T_{\max}$ in order to compensate for the energy consumed in the process of initial heating, $(1 - \eta_t)w_t = (1 - \eta_t)3T$ in (14), contrasting with the corresponding term, $(1 - \eta_t)\dfrac{dw_t}{dt} = 0$ at $\tau > 0$ in (13).

(c) $\bar{\tau}_{\min}$ *and* LAWSON's *criterion*

$\bar{\tau}_{\min}$ is the minimum duration necessary to produce enough reactions at a given temperature to regain the energy consumed in the initial heating process. The tangential line to the contour parallel to T-axis corresponds to LAWSON's criterion. Our $\bar{\tau}_{\min}$ differs from that of LAWSON however in that it is expressed as a function of temperature and includes the effects of P_{bee} but not of P_n, i.e. $\eta_n = 0$. For $n_{D_0} = n_{T_0} = \frac{1}{2}$, the minimum value of $\tau_{\min}$ is $4{\cdot}2 \times 10^{14}$ sec cm^{-3} at $T = 1{\cdot}8 \times 10^{8}$°K with $\eta_e = \eta_b = \eta_t = \frac{1}{3}$ and $\eta_n = 0$ as seen in Fig. 5, (II). On the other hand when $n_{D0} = 1$ $\tau_{\min}$ cannot exist in both cases of $\eta_e = 1/2$ and $\eta_e = 1/3$, as seen in Fig. 6(b), although LAWSON's minimum duration is 10^{16} sec cm^{-3} when $\eta_e = 1/3$. The reason for this difference is mainly attributed to bremsstrahlung caused by electron-electron scatterings.

(d) *The meaning of* $\tau_{\min}$ *and* $\tau_{\max}$

$\tau_{\max}$, the upper limit to τ which satisfies the energy balancing (11) in the narrow sense, exists because of the decrease of unburned fuels and the increase of multiply-charged nuclei in the ashes.

The lower limit $\tau_{\min}$ corresponds to the critical reduced time after which the energy balancing in the narrow sense is achieved, due to the increase of tritium and ^{3}He through DD reactions.

(e) *Application of* $\tau_{\max}$ *to the stationary flow of fuels*

$\tau_{\max}$ may be regarded as the critical time after which the old fuel must be replaced. Thus in reactors

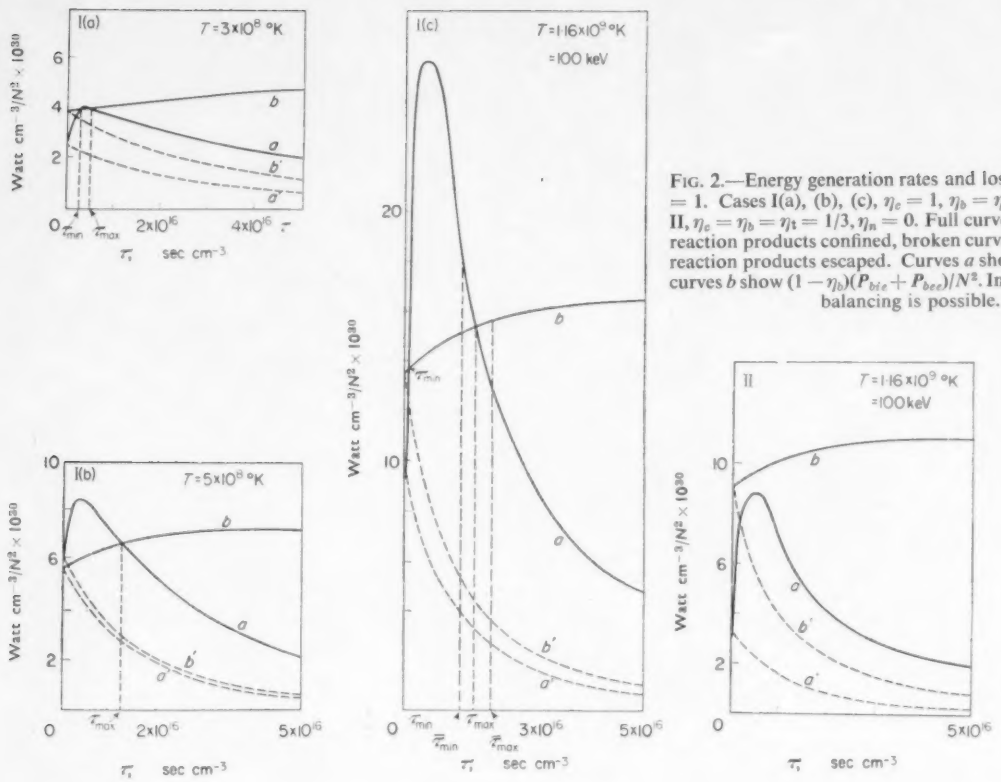

FIG. 2.—Energy generation rates and losses against τ, $n_{D_0} = 1$. Cases I(a), (b), (c), $\eta_e = 1$, $\eta_b = \eta_{\dagger} = \eta_n = 0$. Case II, $\eta_e = \eta_b = \eta_{\dagger} = 1/3$, $\eta_n = 0$. Full curves denote charged reaction products confined, broken curves denote charged reaction products escaped. Curves *a* show $\eta_e(E - P_n)/N^2$, curves *b* show $(1 - \eta_b)(P_{bie} + P_{bee})/N^2$. In Case II no energy balancing is possible.

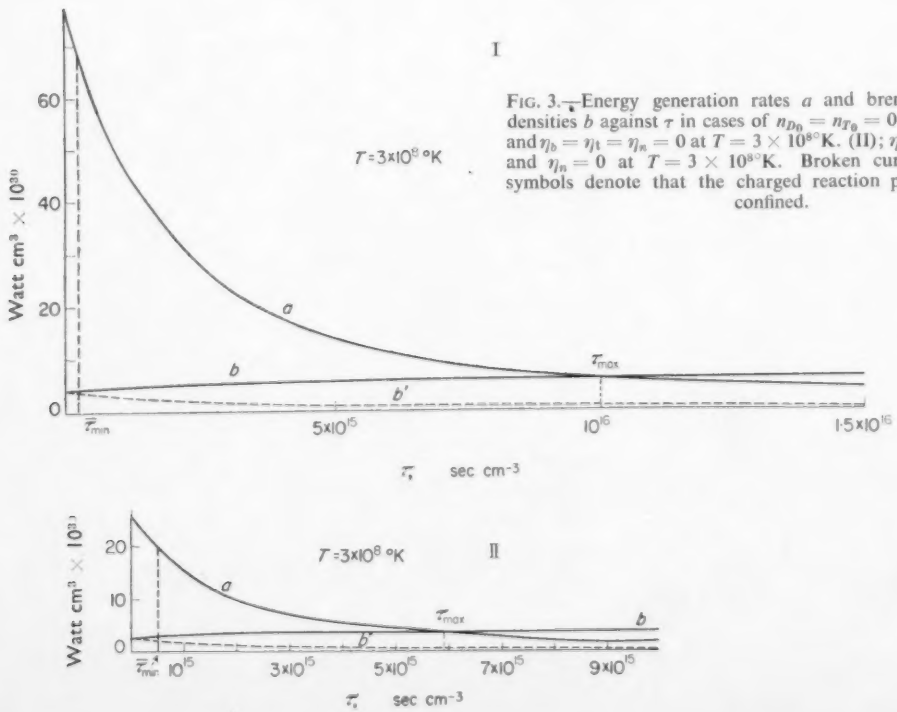

FIG. 3.—Energy generation rates *a* and bremsstrahlung loss densities *b* against τ in cases of $n_{D_0} = n_{T_0} = 0{\cdot}50$. (I); $\eta_e = 1$ and $\eta_b = \eta_{\dagger} = \eta_n = 0$ at $T = 3 \times 10^8$°K. (II); $\eta_e = \eta_b = \eta_{\dagger} = 1/3$ and $\eta_n = 0$ at $T = 3 \times 10^8$°K. Broken curves with dashed symbols denote that the charged reaction products are not confined.

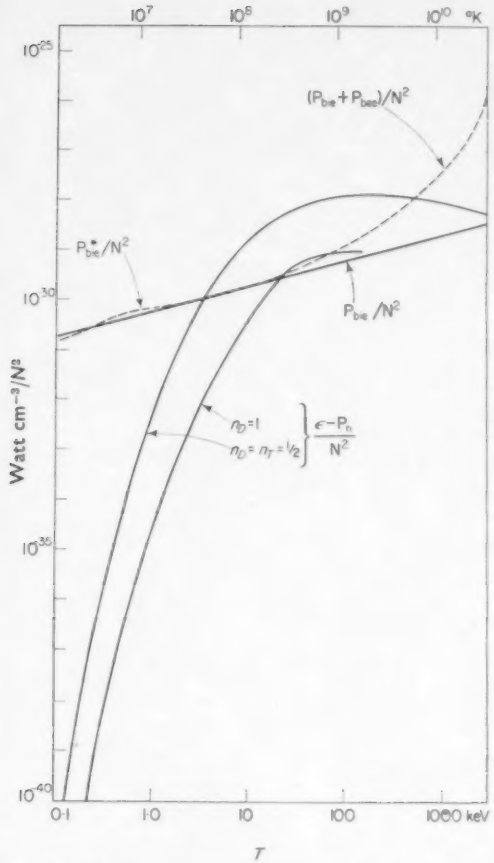

FIG. 4.—Energy generation rates and bremsstrahlung loss densities against T at $\tau = 0$, $\eta_e = 1$, $\eta_b = \eta_t = \eta_n = 0$. P^*_{bie} denotes ion-electron bremsstrahlung calculated with correction to the usual Born approximation. (See note added in proof (†) on page 30.)

with steady flow of the fuels, the velocity should be such that any part of the fuel stays in the reactor only for a critical time interval, τ_e. If the old and new fuels are not mixed in the reactor, τ_e corresponds directly to $\tau_{\max}$. If mixing occurs however, τ_e is somewhat different from $\tau_{\max}$. This case has been studied recently by JENSEN *et al.* (1958). Relations between $\tau_{\max}$ and τ_e must be formulated as a function of the mixing ratio of old and new fuels.

(f) *Total available energy* Q_g

With our assumptions the total available energy per one nucleus, Q_g, as defined in Section 4,

$$Q_g = \frac{1}{N_0}\left\{\eta_e\int_0^{t_{\max}} P_e dt - (1-\eta_b)\int_0^{t_{\max}} P_b dt - (1-\eta_t)W_i\right\}$$

$$= \int_{\bar{\tau}_{\min}}^{\tau_{\max}} \frac{\eta_e P_{tc} - (1-\eta_b)P_b}{N^2}\, a\left\{1 - \frac{d\log a}{d\log \tau}\right\} d\tau$$

As shown in Fig. 8, (II), Q_g is a function of temperature. If we denote a half of Q-value for one nuclear reaction by $Q/2$, Q_g is less than about 20 per cent of $Q/2$ in the case of $n_{D0} = 1$ and several per cent in the case of $n_{D0} = n_{T0} = 0{\cdot}5$. The remaining part of Q is carried off by neutrons which escaped, or remains inaccessible in the unburned part of the fuel.

(g) *Ideal temperature* T_1 and $\bar{T}_1$

As seen in Fig. 8, (I), energy generation rate minus radiation loss density at $\tau = 0$ has a maximum at a certain temperature. We name this temperature "ideal temperature in the narrow sense", T_i. T_i is 8×10^8°K for $n_{D0} = n_{T0} = 0{\cdot}5$, and $5{\cdot}4 \times 10^8$°K for $n_{D0} = 1$.

As shown in Fig. 8, (II) total available energy per nucleus Q_g has a steep maximum at a certain temperature. We name it 'the ideal temperature in the wide sense', $\bar{T}_i$. For $n_{D0} = n_{T0} = 0{\cdot}5$, $\bar{T}_i = 2$–3×10^8°K, regardless of the value of the η's. For $n_{D0} = 1$, $\eta_e = 1$, $\eta_b = \eta_t = \eta_n = 0$, $\bar{T}_i$ is 8×10^8°K and 5×10^8°K in cases with confined and no confined ashes, respectively.

Since $\eta_e P_e - (1-\eta_b)P_b < 0$ at $T > T_{\max}$, and, since $\dfrac{dQ_g}{dT} > 0$ at $T < \bar{T}_i$ we speculate that a fusion reactor at $T > T_{\min}$ is easy to run at a temperature between $T_{\max}$ and T_i. But an exact discussion of this tendency must rest on the basis of equation (16).

(h) $\bar{\tau}_{\max}$

This arises for reasons similar to those which determine $\tau_{\max}$. Since $\bar{\tau}_{\max} > \tau_{\max}$ it should certainly not be reached in an economic reactor.

(i) *The energy balancing region in both senses*

The region which satisfies both energy balancing in the narrow and wide senses is hatched by horizontal strips and denoted by the region III in Figs. 5 and 6.

(j) *T–Nt diagram with technical criteria*

Since T and τ are sufficient to describe thermonuclear systems we can also put technical criteria on the *T–Nt* diagram. For example, limitations and criteria for various heating methods are being studied on *T–Nt* diagrams by T. UCHIDA of Nihon University. In the middle of region 1 (Figs. 5 and 6) there is a milestone marking the threshold of thermal reactions, which is indicated by observable number of neutrons of thermonuclear origin. We denote this upper part of this region as I′. Neutrons yields so far reported give no evidence of a thermonuclear origin. Consequently, all fusion researches are now on the way from I to I′, as seen in Figs. 5 and 6. From the viewpoint of engineering, the pair of fundamental

variables T and τ correspond to heating and confinement of plasmas. Therefore, simple pinch experiments climb the diagonal path to reach the region III on T–Nt diagram. However, the confinement in simple pinches is too unstable, and there are

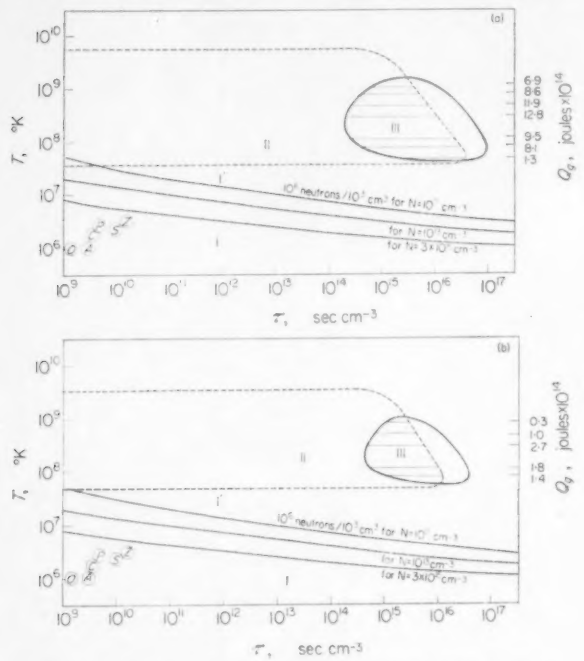

FIG. 5.—T–Nt diagram for $n_{D_0} = n_{T_0} = 0{\cdot}50$ with the assumptions as explained in Section 4 and all charged reaction products confined. (I), $\eta_c = 1$ and $\eta_b = \eta_t = \eta_n = 0$ (II), $\eta_c = \eta_b = \eta_t = 1/3$ and $\eta_n = 0$.

Solid curves are the contours bordering the energy balancing regions in the wide sense, and broken lines are those bordering the energy balancing regions in the narrow sense, (domain II). The regions hatched with horizontal strips are those satisfying energy balancing in the narrow and wide senses, (domain III). The scales on the right ordinate indicate Q_g at corresponding constant temperature on the left ordinate. The relation between temperature and time duration to yield 10^6 neutrons per 10^3 cm^3 is plotted for given N. Points Z, S, C, P, O and A represent the reported experiments of Zeta, Sceptre, Columbus, Perhapsatron, Osaka University and Artsimovich *et al.*

more favourable paths on T–Nt diagram, for instance, heating after confinement and an induced pinch with external field.

6. DEUTERIUM NOT SUFFICIENT AS AN ENERGY SOURCE WITHOUT USE OF NEUTRONS

The balancing region in wide sense for pure deuterium fuels is so small that it quite disappears on T–Nt diagram even if we adopt rather a large value 1/2 for η_c. If we adopt $\eta_c = 1/3$, as the efficiency for usual engines, energy balancing regions both in narrow and wide senses entirely fade out. Since we have taken only the most fundamental losses into account, the power balancing for actual reactors must be more rigorous. This leads to the important

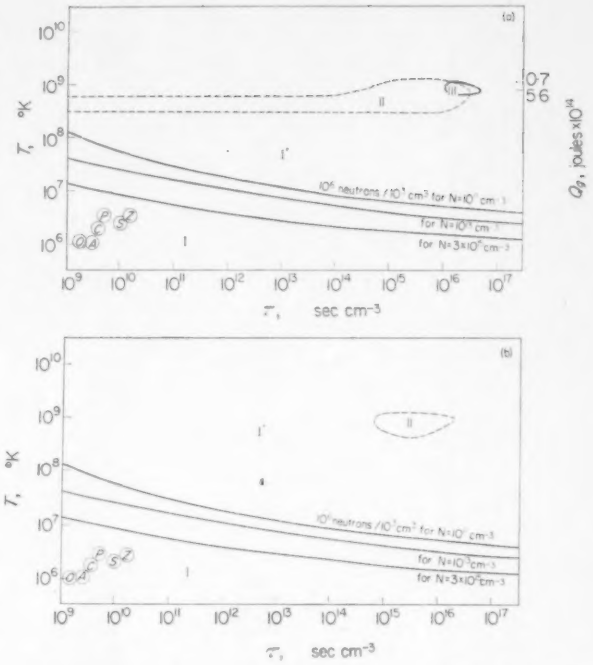

FIG. 6.—T–Nt diagram in the case of confining all charged reaction products for $n_{D_0} = 1{\cdot}0$. Fig. 6(a) is for $\eta_c = 1{\cdot}0$ and $\eta_b = \eta_t = \eta_n = 0$ and Fig. 6(b) for $\eta_c = 1/2$ $\eta_b = \eta_t = 1/3$ and $\eta_n = 0$.

It must be noted that in the case of $\eta_c = 1/2$ there is no region of energy balancing in the wide sense on the T–τ plane. Other explanations are the same as for Fig. 5.

conclusion that pure deuterium fuel seems to be of no practical use as an energy source unless neutrons and their energy are effectively utilized. They can indeed produce so much heat in reactions in which $n_{D_0} \gg n_{T_0}$ that the main contribution to energy

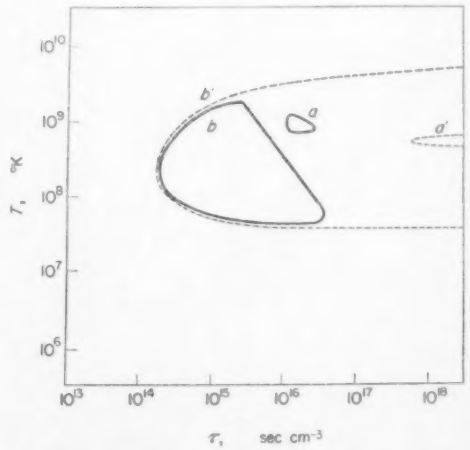

FIG. 7.—The energy balancing in both senses, (domain III), for $\eta_c = 1$ and $\eta_b = \eta_t = \eta_n = 0$. Full curves refer to charged reaction products confined, and broken curves to charged reaction products escaped. a refers to $n_{D_0} = 1$ and b to $n_{D_0} = n_{T_0} = 0{\cdot}5$. It is a remarkable fact that region a is left to a', and on the contrary b is right to b'.

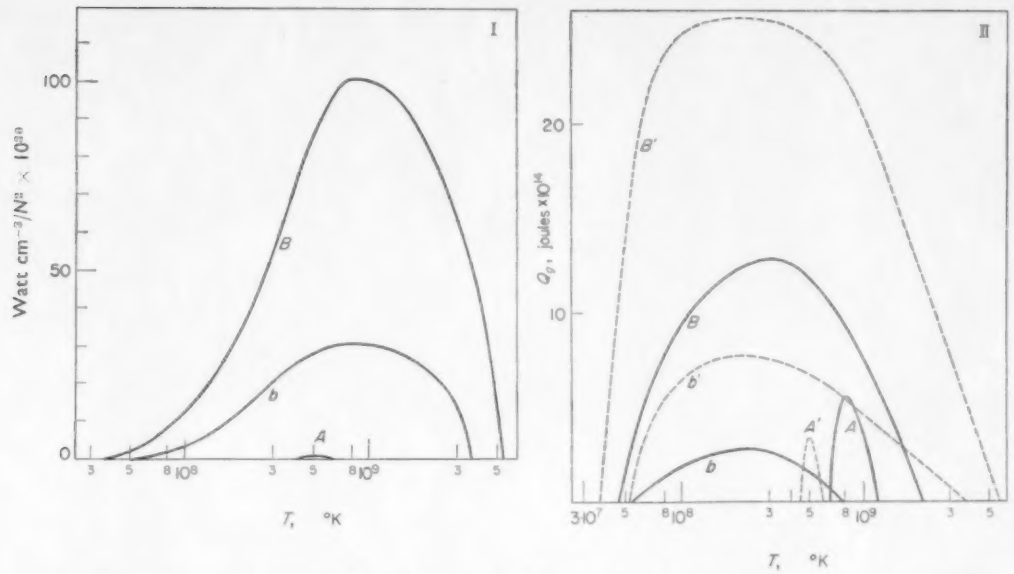

FIG. 8.—(I), Rates of available energy gain, $(\varepsilon - P_n)/N^2$ at $\tau = 0$ and (II), total available energy Q_g as functions of T. A refers to $n_{D_0} = 1$, $\eta_c = 1$, $\eta_b = \eta_t = \eta_n = 0$; B refers to $n_{D_0} = \eta_t \tau_0 = 0{\cdot}5$, $\eta_c = 1$. $\eta_b = \eta_t = \eta_n = 0$; b refers to $\eta_c = \eta_b = \eta_t = 1/3$, $\eta_n = 0$. Broken lines and dashed symbols refer to charged reaction products not confined. It is remarkable (II) that the broken line A' is lower than the solid line A, contrasting with the fact that B' is higher than B.

generation comes from neutrons rather than charged particles. Neutrons may also be used to produce tritium through ^{6}Li(n,t) reactions. Furthermore, some of T and ^{3}He products of the DD reaction remain behind in the remnants after a time $\tau_{\max}$ has elapsed. Pure deuterium fusion reactors may therefore play the role of breeders, and the tritons can be fed back into fusion reactors.

7. THE ROLE OF CHARGED REACTION PRODUCTS AND TIME VARIATIONS OF FUELS

Since the fuels decrease with time the sudden increase of energy generation in Fig. 1 must be attributed to effects of burning some of the secondary nuclei.

The secondary nuclei play two opposite roles in the energy balancing; they increase both the energy generation rate and the radiation losses. In order to illustrate these two roles we compare systems in which all secondary particles escape out of the reacting plasmas, with systems in which they are confined (except neutrons) as discussed before.

The terms including $d \log a$ in (13) and (14) cannot be neglected since the number of ions decreases due to the escape of secondarily produced nuclei. This is so as long as we define $\tau = N(t)t$ and $n_D = N_D(t)/N(t)$. However we can eliminate terms including $d \log a$ by replacing τ and $n_D = N_D(t)/N(t)$ with $\tau' = N_0 t$ and $n_D{}' = N_D(t)/N_0(t)$. This imaginary case gives results the same as that of no confined secondary nuclei, except that $n_D{}'$ is defined as a fraction normalized to $N =$ constant instead of to $N = N(t)$.

Consequently, if $n_\tau = 0$ at $\tau = 0$, $dw_i/d\tau = dw_e/dt = 0$, $w_i = 3T$, $(1 - \eta_c)w_e = 0$ and $d \log a = 0$ as assumed in Section 4,

$$\eta_c(Q - Q_n)_{DD}\langle\sigma v\rangle\frac{n_D{}'^2}{2} + \eta_n Q_{nDD}\frac{\langle\sigma v\rangle_{DD}}{2}n_D{}'^2 - (1 - \eta_b)5{\cdot}4 \times 10^{-31}T^{1/2}\left(1 + \frac{T}{66}\right)n_D{}'^2 \geq 0 \quad (17)$$

$$\int_0^{\tau'}\left\{\eta_c(Q - Q_n)_{DD}\frac{\langle\sigma v\rangle_{DD}}{2}n_D{}'^2 + \eta_n Q_{nDD}\frac{\langle\sigma v\rangle_{DD}}{2}n_D{}'^2 - (1 - \eta_b)5{\cdot}4 \times 10^{-31}T^{1/2}\left(1 + \frac{T}{66}\right)n_D{}'^2\right\}d\tau - (1 - \eta_t)3T \geq 0 \quad (18)$$

$$\frac{dn_D{}'}{d\tau'} = -\frac{\langle\sigma v\rangle_{DD}}{2}n_D{}'^2 \quad (19)$$

We can get the most generous necessary condition with the above assumptions by solving these equations analytically. These results are shown in Fig. 7. In this case, the positive term in (17), or the integrand of the positive term in (18) and also the radiation loss terms are proportional to $n_D{}'^2$. Therefore the ratio of reaction power to radiation loss is not a function of time, and hence there are no $\tau_{\max}$ and $\bar{\tau}_{\max}$ when the secondary nuclei are not confined, as seen in Figs. 2(I) and 7. When the secondary nuclei are confined contributions to the energy generation from D^3He

reactions are a few times larger than those from DD reactions, as in Fig. 1, (b), especially at high temperature.

Consequently when $n_{D0} \gg n_{T0}$, the additional energy obtained from burning the secondary nuclei exceeds the additional loss from radiation, and it is better to confine them as illustrated on Fig. 8, (II).

On the other hand, when $n_{D0} = n_{T0}$, even if all ashes are confined, the reactions TT, D^3He, T^3He, etc. contribute only to smaller order than DT, and it is better to let these ashes escape. This is illustrated on Figs. 7 and 8.

8. CONCLUSIONS

The description of nuclear fusion reactors by two variables T and τ is useful for showing essential limitations and arranging various technical criteria in a unified way. The T–τ plane, or T–Nt diagram makes a strategic map on which energy balancing regions may be marked out. It is also concluded that pure deuterium fuel may be of no practical use as an energy source, except as a breeder, unless we try to utilize neutrons to produce heat through reactions in blankets or associated reactors, or to produce tritium through the (n,t) reactions. An important practical problem therefore is to determine the most economical mixing ratio of deuterium and tritium, the most economical use of neutrons and the most effective production of tritium.

Secondarily produced nuclei have two opposite roles which affect the energy balancing, they can increase the energy generation rates and also the radiation losses. Whether the confinement of secondary particles is better or worse for getting the available energy depends on the initial ratio of deuterium to tritium n_{T0}/n_{D0}; it is better when $n_{D0} \gg n_{T0}$, and worse when $n_{D0} \ll n_{T0}$.

Approximations we have adopted do not spoil the generality of our conclusions about fusion reactors on earth. These approximations are divided roughly into two classes; one consists of approximations appropriate to any thermonuclear systems described in Section 2, and the other consists of those distinguishing reactors on earth from the other nuclear fusion systems, energy generating stars and thermonuclear bombs. In general, the η's for stars and thermonuclear bombs are very near to one. These large efficiencies make energy balancing regions large and easy to reach on T–Nt diagram.

In thermonuclear bombs a high material density, causing an explosive high power density, is used. For this reason it is possible to stop most particles and radiations in the system through the initial stage of explosion which correspond, however, to a reduced time long enough to produce balancing in the wide sense.

Because of their high density and vast size, efficiencies for stars are extremely good. On the other hand, owing to their long lives and high densities, the life of a star is extremely long. Consequently, slower reactions than DD and DT, which cannot have energy balancing regions on T–Nt diagram for reactors on earth, have large energy balancing regions on the T–Nt diagram for stellar interiors. Indeed, proton-proton chains, CH-cycles, α-burning reactions and reactions of other heavier nuclei are most important for stellar energy sources.

On the contrary, such high densities cannot be used for reactors on earth, since η's have low values, and furthermore, stabilization of plasma by the magnetic confinement for a very long time is required in order to satisfy the energy balancing condition $\bar{\tau}_{\min}$ and $\tau_{\max}$.

Finally, we recognize that with fusion reactors on the earth we can take out the energy only from deuterons, tritons, ^{3}He and at most lithium, contrasting with the fact that in the universe even the energy of the most incombustible nuclei is taken out.

Consequently, we should consider fusion reactors in the heavens, (the sun), as well as those on earth as sources of power in the future, particularly for cosmic stations.

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* When preparing this article for publication I learned from abstracts of papers presented at the *Second Geneva Conference on Atomic Energy* that T. H. JENSEN *et. al.*, and Mme. D. MAGNAC-VALETTE *et al.* are studying the energy balancing under isothermal pulsed conditions.

† *Note added in proof:* When this article was in proof I heard from J. STICKFORTH (Private communication) that THOMPSON's value of the cross-section for electron-electron bremsstrahlung is in error. The term proportional to $T^{3/2}$ in $P_b = P_{bie} + P_{bee}$ should be a factor of about two less than that given in Table 1. These effects will be considered in a future paper.

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ON THE CLASSIFICATION OF ELECTRON STREAMS

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Abstract—The relationships between different types of electron stream may conveniently be studied in terms of three dimensionless characteristic parameters. These are f, the ratio of positive to negative charge in the stream, ν, the product of the number of electrons per unit length in the stream and the classical electron radius (e^2/mc^2), and ε defined as $(1-\beta^2)^{-1/2}-1$, where βc is the drift velocity. With the aid of these parameters a number of typical streams in which f is zero (electron beams in vacuum tubes and accelerators) or unity (neutralized electron beams, Budker's self-constricted beam, pinched gas discharge plasmas) are studied, and relationships between them are illustrated on the $\log\varepsilon - \log\nu$ plane.

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1. INTRODUCTION

STREAMS of electrons or of other charged particles are of interest in many branches of physics. Several types of stream, each with different properties, may be distinguished; thus the electron beam in a cathode ray tube behaves differently from a gas discharge or a current in a copper wire. In this paper some fundamental properties of electron streams are discussed with the aid of dimensionless parameters, and the relation between different types of stream is illustrated by means of a chart. In the classification to be attempted, electron streams will be divided into two main classes; un-neutralized streams which consist of electrons only, and neutralized streams in which the electrostatic space charge forces are substantially neutralized by positive ions. The term 'beam' will be used for a stream in which component of electron velocity along the stream exceeds the transverse velocity. The properties of partially neutralized streams will be evident from interpolation between the two extremes.

2. UN-NEUTRALIZED STREAMS

The elementary properties of un-neutralized electron beams are well understood. A good account of these has been given by PIERCE (1949). The flow pattern in such beams is determined by the 'perveance', (defined as $I/V^{3/2}$ amp/(volt)$^{3/2}$ where V is the potential of the beam with respect to the cathode from which it originates), and is independent of the size of the system provided that any external focusing fields which there may be are suitably scaled. These scaling laws are illustrated by the 'universal beam spread curve', which shows the behaviour in the absence of externally applied fields of a beam of uniform cross section in which the electron trajectories are initially parallel; the ordinates and abcissae of this curve (PIERCE, 1949) are proportional to r/r_1, and $(I/V^{3/2})z/r_1$ respectively, where r_1 is the initial beam radius.

In the present paper alternative parameters, which are dimensionless, are used to describe the beam. Thus β denotes the ratio of the mean component of electron velocity along the beam (drift velocity) to the velocity of light, and ν is the product of N, the number of electrons per unit length of the beam, and r_0 the classical electron radius. In terms of these quantities, $I = Ne\beta$, and for a low density non-relativistic electron beam in which the electrons move in substantially parallel orbits, $eV = \frac{1}{2}m\beta^2c^2$. Two other useful parameters are γ_e, which is the mean value of the total energy of the electrons expressed in terms of the rest energy, and γ_d, which is defined as $(1-\beta^2)^{-1/2}$; γ_d may be identified with the ratio of total energy to rest energy of an electron moving with the drift velocity βc. The ratio of total kinetic energy to rest energy for such an electron will be denoted by ε. In terms of β and γ:

$$\left.\begin{aligned}\gamma_d &= \varepsilon + 1\\ \beta &= (\varepsilon^2+2\varepsilon)^{1/2}/(1+\varepsilon)\\ &\to (2\varepsilon)^{1/2} \text{ as } \varepsilon \to 0\\ &\to 1 \text{ as } \varepsilon \to \infty\end{aligned}\right\} \quad (1)$$

The advantage of using these parameters will be evident later, when neutralized beams are considered.

The characteristics of different types of beam may well be exhibited on the $\log\varepsilon - \log\nu$ plane (see Fig. 1). Lines of constant current are given by setting $\nu\beta$ constant; thus if I is in amps, $I = 17{,}000\,\nu\beta$, and this may be plotted directly with the aid of equation (1).

The fundamental limit to electron beams due to

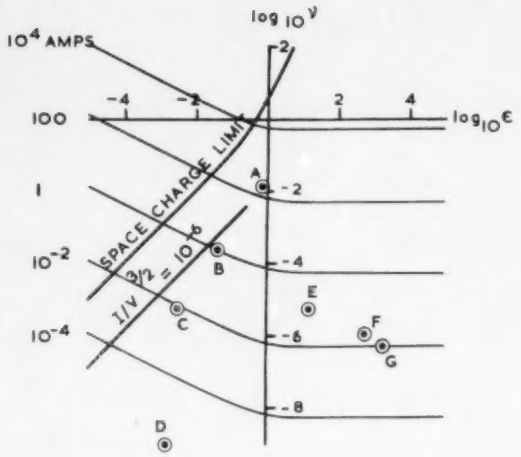

FIG. 1.—Beam chart for un-neutralized electron beams. The $\log \varepsilon = 0$ line divides the chart into non-relativistic and relativistic regions. Electron beams corresponding to points to the left of the space charge limit line cannot exist, and the $I/V^{3/2} = 10^{-6}$ line forms a practical limit. Points corresponding to a number of actual electron beams are marked, and brief identification is given below.

A—High power pulsed klystron (Stanford, 1954)
B—C.W. X-band klystron (Ferranti, 1956)
E—Low power reflex klystron (CV35)
D—Typical cathode ray tube
E—8 MeV medical linear electron accelerator (Metropolitan-Vickers, 1953)
F—Circulating current, Glasgow synchrotron
G—Circulating current, Cornell synchrotron.

space charge may conveniently be shown as a line on this 'beam chart'. A suitable expression for this will now be derived. The spreading of a cylindrical beam of uniform cross section under its own space charge is given in these units by, (LAWSON, 1958),

$$rd^2r/dz^2 = 2\nu/\beta^2\gamma_e{}^3. \tag{2}$$

Now it has been shown by HARRISON (1958) that for small values of z the solution of this equation is a hyperbola passing through $r = r_1$, $z = 0$, with asymptotes at an angle to the z-axis given by

$$\tan\theta = (2\nu/\beta^2\gamma_e{}^3)^{1/2}. \tag{3}$$

For a beam to exist, therefore, $\tan\theta$ must be much less than unity, or

$$\nu \ll \tfrac{1}{2}\beta^2\gamma_e{}^3. \tag{4}$$

For a non-relativistic beam, $\gamma = 1$, $\nu\beta \propto I$, $\beta^2 \propto V$, and equation (3) may be written

$$I/V^{3/2} \ll 68 \times 10^{-6} \text{ amp/(volt)}^{3/2}. \tag{5}$$

This is well known to designers of klystrons and travelling wave tubes, who find that $I/V^{3/2} \leqslant 2 \cdot 10^{-6}$ is a good practical rule. It is interesting to note that the condition for the theoretical maximum current obtainable with Brillouin flow, given by PIERCE (1949) as

$$I/V^{3/2} = 25{\cdot}4 \times 10^{-6} \text{ amp/(volt)}^{3/2} \tag{6}$$

becomes

$$\nu = \beta^2 = 2\varepsilon, \qquad (\gamma_e \sim 1) \tag{7}$$

when expressed in the units used here. All electrons except those on the axis move in spiral orbits, and the outermost electrons have a circumferential velocity equal to $\sqrt{2}\beta c$.

The inequality (4) is also well illustrated by the expression for the maximum charge which can be held in a betatron field. If n, the field index, (defined as $-rdB/Bdr$), is equal to 0·5, then the number of electrons per unit length which can be held in a toroidal tube with small and large radii r and R respectively is given by

$$\nu = \tfrac{1}{4}\beta^2\gamma_e{}^3(r/R)^2, \quad \text{where } r \ll R. \tag{8}$$

On the basis of equations (3), (7) and (8), a rough 'space charge limit' curve may be defined as

$$\nu = \tfrac{1}{2}\beta^2\gamma_d{}^3$$
$$= \tfrac{1}{2}\varepsilon(\varepsilon + 1)(\varepsilon + 2). \tag{9}$$

γ_d has been written here for γ_e which occurs in equations (4) and (8). The relation between them is not well defined; they are equal in the betatron, but in the case of the spreading beam γ_d decreases as the beam spreads. In the non-relativistic region, this distinction does not matter since $\gamma_e = \gamma_d = 1$. The space charge limit curve is shown on Fig. 1; all points corresponding to electron beams lie to the right of it. Any electron beam may be represented as a point on the chart, though such a point does not entirely characterize the beam. Points corresponding to beams in some typical devices are shown.

3. NEUTRALIZED SYSTEMS WITH NO COLLISIONS

Neutralization of the space charge of an electron stream suppresses the repulsive force between the electrons, and only the attractive magnetic force remains. In an important class of streams it is this magnetic force which holds the stream together; such 'self-constricted' streams tend in practice to be extremely unstable, but they are nevertheless of considerable academic interest.

Self-constricted streams were first studied by BENNETT (1934). In his paper collisions are neglected and electrons and ions move in opposite directions with a relative velocity much greater than the transverse velocity. The transverse velocity distribution is

taken as Maxwellian, so that a two-dimensional temperature can be defined. Each component of the stream is acted upon by a common radial electrostatic field, which acts in opposite directions on the two components, and a magnetic field which produces a force inwards in both cases. For the special case in which the ion drift velocity is zero, the following important relation holds

$$I^2 = 2Nk(T_i + T_e). \tag{10}$$

T refers here to a 'transverse' temperature, which for electrons may be written $\frac{1}{2}\gamma_e m\langle\beta_t\rangle^2 c^2$. If T_i is zero, and I is set equal to $Ne\beta$, equation (10) becomes

$$\langle\beta_t^2\rangle/\beta^2 = \nu/\gamma_e. \tag{11}$$

This expression is analogous to the 'perveance' relation for charged streams (LAWSON, 1958). It has been assumed so far that $\beta_t \ll \beta$, and equation (11) shows that this implies that $\nu \ll \gamma_e$. Physically this criterion states that the Larmor radius of the electron in the self field is much greater than the beam radius. Nevertheless it may be shown that equations (10) and (11) are true even if $\nu \gg \gamma_e$, (POST, 1956), in which case T represents a true three dimensional temperature. The transition which occurrs at $\nu = \gamma_e$ is very important. For $\nu < \gamma_e$ streaming motion predominates and the system is an electron beam, for $\nu > \gamma_e$ transverse velocities greatly exceed the drift velocity, and the system may be termed a plasma stream.

It is of interest to draw a beam chart similar to Fig. 1 for neutralized streams. Instead of the 'space charge limit' curve, the 'neutral transition' curve given by $\nu = \gamma_e$ may be plotted. Before doing this, however, the relation between γ_e and γ_d must be determined. This is in general a complicated function of ν and also of the beam structure. Since confusion between these quantities has sometimes occurred, the relation between them is discussed in detail in the appendix, where the following approximate expression, valid when $\nu \leqslant \gamma_e$, is derived:

$$\gamma_d = \{(\gamma_e^2 + \gamma_e\nu)/(1 + \gamma_e\nu)\}^{1/2}. \tag{12}$$

Expressing γ_d in terms of ε, and eliminating γ_e between (12) and the expression for the neutral transition curve, $\nu = \gamma_e$, the latter may be written

$$\nu = (\varepsilon + 1)/(1 - 2\varepsilon - \varepsilon^2)^{1/2}. \tag{13}$$

To the left of this line are plasmas, to the right are neutralized electron beams (Fig. 2).

In the non-relativistic region, lines of constant electron temperature can be put on the chart. From equations (1) and (10), with $\beta \ll 1$,

$$\varepsilon\nu = kT/mc^2. \tag{14a}$$

So far, collisions have been neglected; in the next section two important types of stream, in which both collisions and a longitudinal electric field are important, will be discussed.

4. SELF-CONSTRICTED STREAMS WITH COLLISIONS

In order to maintain the current in streams in which collisions are important, an electric field must be applied along the length of the stream. Furthermore, for a steady state solution to be possible, some form of energy dissipation is necessary, so that particles may lose the energy acquired from the electric field. In a self-constricted stream there are no wall losses, so that if a steady state is possible

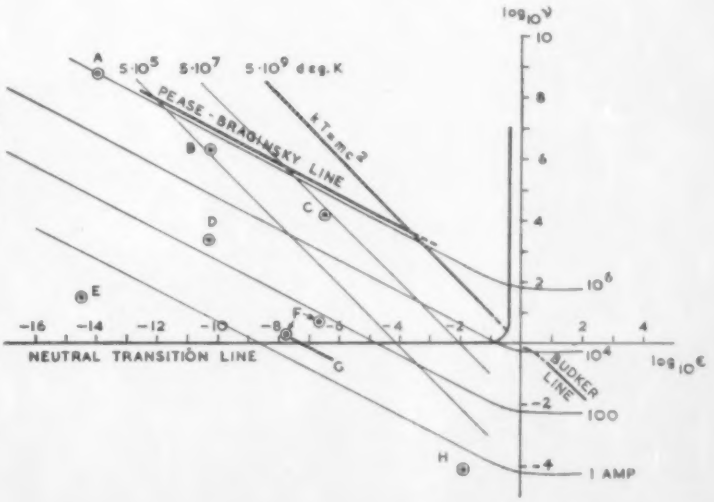

FIG. 2.—Beam chart for neutralized electron beams.

The chart is divided into two by the neutral transition line, to the left of this line random velocities exceed the drift velocity, and to the right of it streaming motion is possible, though not inevitable unless the beam is self-constricted. The temperatures marked in the second quadrant apply only to self-constricted beams in the absence of a longitudinal magnetic field. Points corresponding to a number of typical plasma streams are given below.

A—Exploding copper wire (16 gauge)
B—High power spark (Allen and Craggs, 1954)
C—Zeta, Sceptre and Perhapsatron S3 (1958)
D—Low power spark (Craggs and Meek, 1947)
E—Luminous neon advertising tube
F—Low pressure mercury arc (Thonemann and Cowhig, 1951)
G—Low pressure mercury arc (Killian, 1930)
H—Neutralized electron beam (Field, Spangenberg and Helm, 1947).

energy must be lost by radiation. Two types of radiation are possible in a completely ionized system; these are bremsstrahlung, from the acceleration of electrons by the electric fields associated with the ions, and synchrotron radiation due to the curvature of the electron orbits in the self-magnetic fields. It turns out that bremsstrahlung is important when $\nu \gg 1$, $\varepsilon \ll 1$, and synchrotron radiation when $\nu \ll 1$, $\varepsilon \gg 1$. The first case has been considered by PEASE (1957) and by BRAGINSKY (1958) and the second by BUDKER (1956a,b).

The method of finding the steady state is the same in both cases; energy and momentum balance along the stream are considered. Electrons are assumed to lose momentum, but no energy, in Coulomb collisions with the ions, and to lose energy (and momentum in one case) by radiation. The rate of change of momentum per electron is Ee and the rate of gain of energy from the electric field is $Ee\beta c$. Elimination of the electric field between the energy and momentum equations, and the use of equations (11) and (12) yield essentially the following results, which, it should be noted, are independent of beam radius:

$$\nu\beta = (3\pi L/2\alpha)^{1/2}, \qquad (\beta \ll 1, \nu \gg 1) \qquad (14b)$$

$$\nu\gamma_f = C, \text{ where } C + C^4 = L, \quad (\gamma_f \gg 1) \qquad (14c)$$

where L is the coulomb scattering logarithm (SPITZER, 1956) and α is the fine structure constant. Numerically these may be written in terms of current as

$$I = 1{\cdot}4 \times 10^6 \text{ amp} \qquad (15a)$$

$$I = 3 \times 10^4/(\varepsilon + 1) \text{ amp} \qquad (15b)$$

respectively. The values of L used are those taken by BRAGINSKY and BUDKER respectively.

These are shown in the chart as the "Pease–Braginsky line" and the "Budker line". In plotting the "Budker line" the distinction between γ_e and γ_f has been ignored, this is unfortunate but necessary if simplicity is to be preserved.

It must be emphasized that there are many assumptions involved calculating these 'steady states', which certainly break down as ν and ε approach unity from both sides of the origin. There is probably no steady state near the origin, but if there is, it will almost certainly consist of at least two co-existing streams, one with low energy and $\beta \ll 1$, and the other with higher energy and $\beta \sim 1$.

These streams both suffer from 'kink' instability, and their practical realization seems remote at the present time.

5. PRACTICAL STREAMS

Actual electron streams are generally more complicated than those discussed in Section 4. For example, both the boundaries of the system and the presence of neutral atoms can have a strong influence on the properties of the stream. Furthermore, radial electrostatic fields can exist in high density streams ($\nu \gg 1$) when the condition of true neutrality is only slightly upset. Nevertheless it is of interest to plot points corresponding to some typical streams on the chart. The temperature calculated for self-constricted streams is not applicable if wall effects are important though it is often of the correct order of magnitude for high current discharges.

6. CONCLUSION

Relationships between electron beams and plasma streams may be brought out clearly by using the dimensionless parameters ν, proportional to the number of electrons/unit length, and ε, which is the ratio of the kinetic energy to the total energy of an electron moving with the drift velocity. If different types of stream are plotted in the $\log \varepsilon - \log \nu$ plane it is found that the type of stream is very roughly characterized by its angular position from the origin ($\varepsilon = \nu = 1$). In particular, the line $\log \varepsilon + \log \nu = 0$ forms a demarcation line between experimentally accessible points (to the left) and points which seem to be inaccessible. Almost on the demarcation line are the BUDKER (1956b) proposed accelerator ($\varepsilon = 2$, $\nu = 0{\cdot}6$) and a fusion reactor based on the pinch effect, ($\log_{10} \varepsilon \sim -5$, $\log_{10} \nu \sim 4$).

APPENDIX

On the relation between γ_d and γ_e

It is possible to define three significant velocities associated with the longitudinal motion of the electrons. When $\nu \ll 1$, $\beta \ll 1$ these are all equal, but failure to distinguish between them in the relativistic region can lead to confusion. These are (normalized with respect to c):

β_d = drift velocity of electrons, defined as the average component of velocity along the stream as seen by an observer at rest

β_e = speed of an electron

β_f = velocity of that frame of reference with respect to which the mean electron velocity is zero. This differs from β_d only in the relativistic region, where the linear law of addition of velocities no longer holds.

Associated with these are values of γ defined as $\gamma = (1 - \beta^2)^{-1/2}$. There is the additional relation also

$$\gamma_e = \text{total energy of electron/rest energy.}$$

The forms of the definition of γ shows that in the relativistic region the values of β may all be very near unity, but that the values of γ may be different, thus changing β from 0·98 to 0·99 changes γ by a factor of $\sqrt{2}$.

In order that the results of this and previous papers can be kept simple, identification of different β's has had to be made. Thus, in LAWSON (1958) no distinction between β_e and β_d has been made; this is a reasonable approximation when $\nu \ll \gamma$. When β approaches unity, however, γ_e is not necessarily equal to γ_d, thus γ has been defined in the above paper as γ_d, whereas it should be γ_e. Similarly in parts of BUDKER (1956b) γ_f has been misinterpreted as γ_e; the relation between them depends on the beam structure, and in his case appears to be

$$\gamma_e = \gamma_f(1 + \gamma_f\nu). \tag{A1}$$

It is necessary for the present paper to find a relation between γ_d and γ_e. As with the relation between γ_e and γ_f this depends on the detailed structure of the beam. The following argument yields an order of magnitude relation. The average inclination of an electron trajectory to the axis is of order β_t/β_d which from equation (11) may be written $(\nu/\gamma_e)^{1/2}$. Consequently

$$\beta_e = (\beta_t^2 + \beta_d^2)^{1/2} = \beta_d(1 + \nu/\gamma_e)^{1/2}. \tag{A2}$$

Replacing β by γ this becomes

$$\gamma_d^2 = (\gamma_e^2 + \gamma_e\nu)/(1 + \gamma_e\nu), \tag{A3}$$

which is the relation required.

LIST OF SYMBOLS

B = magnetic field (e.m.u.)
c = velocity of light
e = electronic charge (e.s.u.)
E = electric field (e.s.u.)
I = current (e.m.u., unless stated to be in amperes)
k = Boltzmann's constant
L = Coulomb logarithm
m = electron rest mass
N = number of electrons per unit length of stream
r = radial co-ordinate
T = temperature (°K)
V = electrostatic potential (e.s.u., unless stated to be in volts)
z = axial co-ordinate
α = fine structure constant (1/137)
β = mean drift velocity of electrons/c
β_f = velocity/c of frame of reference in which the mean electron velocity is zero, equal to β in non-relativistic region
β_t = transverse velocity/c
γ_e = ratio of total energy of electron to rest energy
$\gamma_d = (1 - \beta^2)^{-1/2}$
$\gamma_f = (1 - \beta_f^2)^{-1/2}$
$\varepsilon = \gamma_d - 1$ = ratio of kinetic energy to rest energy of an electron moving with a velocity equal to the drift velocity
$\nu = Ne^2/mc^2$

Subscripts e and i refer to electrons and ions, though where there is unlikely to be confusion the subscript e is omitted.

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A PLASMA BETATRON*

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Abstract—An apparatus is described in which betatron acceleration is applied to electrons in a plasma. A runaway electron current of 1 A at an energy of 2 MeV was obtained.

INTEREST in self-constricting, circulating, high current beams of relativistic electrons has recently been intensified following a proposal by BUDKER (1956) to use the very strong magnetic fields within such a beam for focusing additional ions while they are being accelerated to high energies. One way to consider setting up such a beam is to apply betatron acceleration to the electrons in a plasma, using an electric field which is sufficiently strong to decouple the electrons from the plasma and cause them to be accelerated continuously in the direction of the field.

This process has been discussed by several authors (WILSON, 1925; GIOVANELLI, 1949; DREICER, 1957; LINHART, 1957; DREICER, 1958; GIBSON, 1957), and relatively low energy X-rays associated with runaway electrons have been observed by many experimenters. BERNSTEIN *et al.* (1958), for instance, has obtained runaway electrons of energies up to 2 MeV in a stellarator. A design for a "Megatron" aimed at producing a high intensity 100 MeV electron beam has been given by FINKELSTEIN (1958). It was in order to study this runaway process, and to determine what high energy electron beam intensity could be obtained by betatron acceleration of electrons in a plasma, that the work described here was begun.

Two plasma betatrons have been constructed and operated, the chief difference between them being the higher accelerating electric field afforded by the second machine. Only relatively low energy runaway electrons were obtained with the first machine, which was used to study in some detail, the production of runaway electrons as well as their loss relatively early in the acceleration period. This work has been described in detail in a C.E.R.N. report (REYNOLDS and SKARSGARD, 1958). Only the betatron with the higher accelerating electric field will be described here.

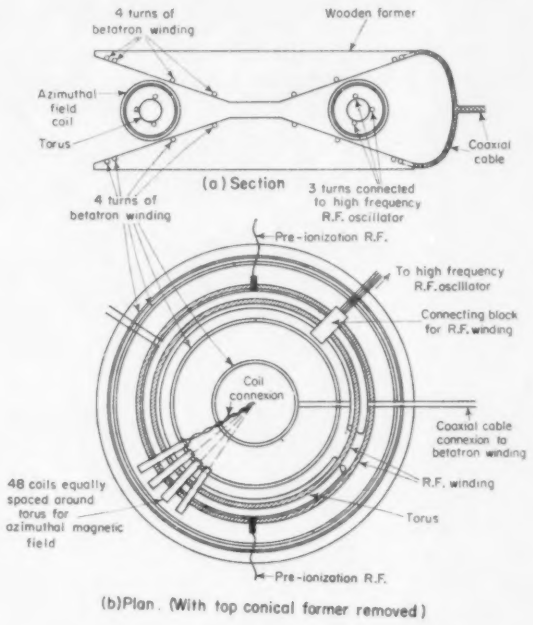

FIG. 1.—Sectional and plan views of plasma betatron.

Figs. 1(a) and 1(b) show the construction of the plasma betatron. A plasma is formed by striking an r.f. discharge in gas at low pressure contained in the toroidal ring chamber (torus) of 52 cm diameter and 5 cm bore. The r.f. is pulsed for a period of between 50 and 500 μsec. Simultaneously with switching off the r.f. power, betatron acceleration is begun by connecting a 2·75 μF bank of condensers charged to high voltage across the air-cored betatron winding (Fig. 2).

The betatron accelerating field is provided by two air-cored half-windings, one placed above, and the other below the plane of the orbit, and supported by conical wooden formers as shown in Fig. 1. Each half-winding consists of four turns in series—connected by radial segments, made co-axial with the return current lead, in order to minimize azimuthal inhomogeneities in the betatron magnetic guiding field. The two half-windings are connected in parallel

* The work described was done in the Accelerator Research Group of C.E.R.N. during the period 1957–8. J. G. LINHART of C.E.R.N. and E. R. EMIGH on leave of absence from Los Alamos Research Laboratory, New Mexico participated in some of the work acknowledged at the end of the paper.

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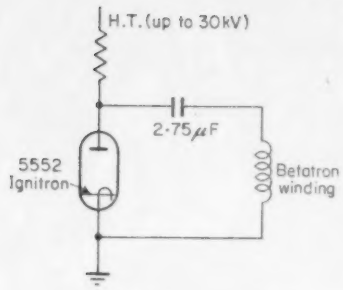

FIG. 2.—Betatron circuit.

in order to provide a higher accelerating field. For the same reason a small number of turns was used for each half-winding. The resulting focusing field about the equilibrium orbit of radius of 26·6 cm consequently is impaired. In the radial direction, the field gradient n varies from 0·3 at the equilibrium orbit to 1·2 at the walls. Within small regions near the top and bottom extremities of the torus cross-section there is a vertical defocusing force. When the condensers in the betatron circuit are initially charged to 25 KV the maximum accelerating field at the orbit is 18 volt/cm; at the end of the first quarter cycle of voltage, the betatron guiding field at the orbit reaches 315 gauss, corresponding to an electron energy of 2·0 MeV. Fig. 3(a) shows the waveform of the voltage during the acceleration period in which there is a transient rise of duration 1 μsec, caused by the time required for the ignitron switch to close. As used for the present work, the current in the betatron windings is allowed to ring, giving a slowly damped 32 kc/s oscillation lasting for about 1 msec.

Plasma is formed in the torus at gas pressures down to 5×10^{-5} mm Hg by means of two r.f. oscillators. Pre-ionization is produced by applying a voltage of approximately 1·5 kV at 5 Mc/s from one oscillator, across two electrodes at diametrically opposite points of the torus. The electrodes (Fig. 1b) are connected to a thin layer of graphite which covers the inner surface of the pyrex glass torus, and which also serves as an electrostatic screen. The resistance of this graphite turn is 8 kΩ. Following the pre-ionization, up to 2–3 kW of r.f. power can be transformer-coupled into the plasma turn from a high voltage oscillator connected to the three-turn primary winding on the outer wall of the torus. A peak voltage of up to 2 kV at 4 Mc/s thus can be induced around the plasma turn. The anode voltage for the driving stage of the high voltage oscillator is supplied from a reservoir condenser which gives a decaying voltage with time, and the amplitude of the r.f. voltage coupled to the plasma turn consequently also diminishes with time. The equilibrium between plasma loss and formation results in a lower plasma density as the r.f. amplitude diminishes. Betatron acceleration is started at the end of the r.f. pulse and therefore, by adjusting the duration of this pulse, the initial plasma density can be closely controlled.

For the duration of the plasma formation and the betatron acceleration there is a steady azimuthal magnetic field of up to 1500 G, provided by 48 closely spaced coils about the aximuth (Fig. 1b), and parallel connected to the centre of the machine. These coils are supplied with an approximately rectangular pulse of current of 15 msec duration from a three-section lumped-element delay line. This is matched to the resistance of 0·5 Ω of the parallel combination of coils. Electrolytic condensers charged to 400 volts are used and the 48 coils form the inductance of the last section of the line as well as the matched resistive load. The time sequence of processes involved in one complete operation of the plasma betatron is controlled electronically by the use of conventional type trigger and delay circuits. Up to now no attempt has been made to put the betatron on an efficient duty cycle. Charging time for the condensers in the betatron circuit and the delay line pulses are several seconds.

The circulating current in the torus during betatron acceleration is measured by means of a solenoidal coil wound on a flexible cylindrical former which is wrapped around the cross-section of the torus. The voltage signal from this coil proportional to di/dt is integrated to give the current waveform. A lower limit of current measurement of 5–10 A was set by a voltage induced in the coil by the flux of the betatron guide field. X-rays are detected by means of a plastic scintillation crystal mounted on a photo-multiplier and placed near the torus. A platinum ring target for the electrons intrudes about 3 mm from the inside surface of the torus.

Runaway electrons have been obtained using the following gases in the torus: hydrogen, nitrogen, oxygen, argon, mercury vapour and xenon. The minimum pressure at which an r.f. discharge could be struck decreased, and the X-ray intensity increased in this order, by about a factor of 10. Runaway electrons have been observed accompanied by conduction currents of up to several hundred amps peak, as well as for conduction currents below the limit of measurement. The conduction current lasts for from 1 to 2 μsec, and its magnitude is controlled directly by the initial plasma density. Its extinction shortly after the start of betatron acceleration is due to the deflecting

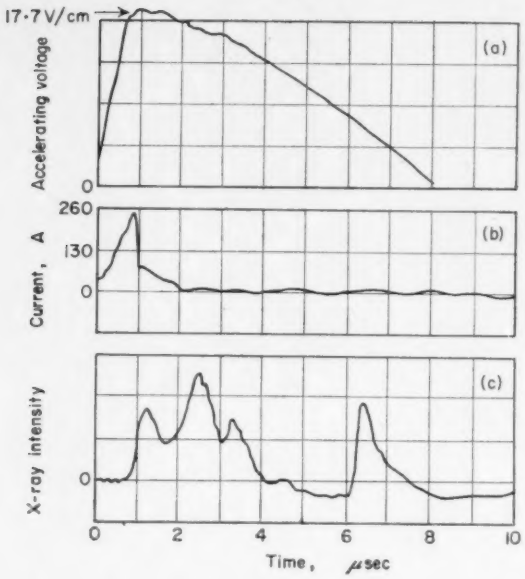

FIG. 3.—Voltage, conduction current and X-ray intensity for xenon at $p = 6 \times 10^{-5}$ mm Hg, and $B_\phi = 1400$ G.

action of the growing betatron guide field on the relatively slow-moving electrons making up the conduction current, which drives the discharge to the inner wall of the torus.

The maximum total number of runaway electrons is obtained for initial plasma densities giving conduction currents of a few hundred amperes peak. Large X-ray pulses are observed from 1 to 7 μsec after the beginning of betatron acceleration, corresponding to runaway electrons of energy 50 keV to 2 MeV. Most of the runaway electrons are lost to the target at 1 to 2 μsec when the conduction current is falling rapidly. (This could be due to the combined effects of back e.m.f. and the self-magnetic field of the conduction current, which are in the right direction and of the correct order of magnitude to produce the losses observed (REYNOLDS and SKARSGARD, 1958). The energy corresponding to this time is about 100 keV. The intensity of X-rays indicates that the circulating current represented by these 100 keV electrons is up to 20 A. Accelerating voltage, current and X-ray records for a typical discharge in xenon are shown in Figs. 3(a), 3(b) and 3(c).

For lower initial plasma densities, giving conduction currents less than the 5–10 A limit of measurement, X-rays can be observed at times near the end of the acceleration period. Lead absorption measurements on these X-rays indicate that their energy corresponds to runaway electrons with nearly the full energy of the machine, which for an initial condenser voltage of 25 kV is approximately 2 MeV. The circulating current

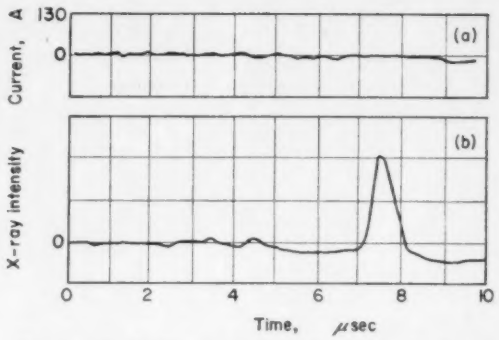

FIG. 4.—Conduction current and X-ray intensity for argon at $p = 1 \times 10^{-4}$ mm Hg, $B_\phi = 1400$ G and $E_{max} = 17{\cdot}7$ V/cm.

represented by these electrons is estimated from the X-ray intensity to be about 1 A. Figs. 4(a) and 4(b) show current and X-ray records for such a discharge in argon. The disturbing effects of the conduction current on the particle trajectories is smaller than in the previous case and it is to be noted that now X-rays are not produced early in the acceleration process.

The production of high energy X-rays depends strongly on the accelerating voltage. If the initial condenser voltage is decreased to 10 kV only X-rays corresponding to about 100 keV electrons are observed. The intensity of these low energy X-rays is about an order of magnitude lower than similar energy X-rays obtainable at 25 kV. Theories of the runaway process are in qualitative agreement with this observation (DREICER, 1957; LINHART, 1957; DREICER, 1958).

No X-rays have been observed with zero azimuthal magnetic field, even though it was still possible to regulate the initial plasma density to give conduction currents of up to several hundred amperes peak. For B_ϕ equal to 300 G small intensities of X-rays corresponding to low energy electrons were observed. In general, the intensity of high energy X-rays increases with the azimuthal magnetic field up to the maximum fields used of about 1500 G. Presumably, B_ϕ is important initially in holding the electrons, of non-zero starting energy, until they are caught in an equilibrium orbit. Losses of runaway electrons to the torus walls, through disturbing electromagnetic effects accompanying the conduction currents, for example, also are slowed down by the presence of B_ϕ.

The variation of X-ray intensity with pressure in the vicinity of 1×10^{-4} mm Hg is not strong. It becomes important however, for pressures about an order of magnitude higher; the X-rays intensity falls with increase of pressure.

The production of high energy X-rays depends

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rather sensitively on the quality of the betatron field. Azimuthal inhomogeneities of a few per cent in the betatron guiding field are sufficient to destroy the high energy electron beam completely. The machine is provided with a beam knockout facility. The betatron field is disturbed at any desired time during the acceleration period by short-circuiting part of the winding. However, beam knockout has not been found to increase the intensity of high energy X-rays. Apparently all of the runaway electrons are lost when the accelerating field approaches zero. This may be due to a resonance loss mechanism which becomes important as the ratio of the betatron guiding field B_z, to the azimuthal magnetic field B_ϕ approaches a slowly-changing value.

The results observed with this plasma betatron suggest that, if higher accelerating voltages were provided, much higher circulating currents of full energy runaway electrons could be expected. Also for high accelerating fields the necessity for a strong azimuthal magnetic field—harmful due to its suppression of beam constriction—may be less important. Electrons should be caught in an equilibrium orbit before they have time initially to drift from the focusing region. The maximum runaway electron current obtainable in a plasma betatron taking into account the disturbance of the magnetic field pattern produced by the electron current, has recently been calculated by MAISONNIER and FINKELSTEIN (1959). For this betatron the current is 350 A, but a design is proposed which might enable higher currents to be obtained. There seems, therefore, to be no fundamental difficulty in establishing within a plasma high energy electron currents of interesting magnitude. The attainment of the very intense currents demanded by the BUDKER (1956) proposal may require the overcoming of instabilities of various types peculiar to self-focusing relativistic electron streams (BUDKER, 1956; BUNEMAN, 1958; SCHOCH, 1958; FINKELSTEIN and STURROCK, 1959).

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ON THE CONFINEMENT OF CHARGED PARTICLES IN A MAGNETIC FIELD

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Abstract—A self-sustained thermonuclear fusion reaction requires a high degree of particle confinement. A ring-shaped *solid* conductor placed in the interior of the reaction volume produces a strong poloidal magnetic field which traps particles completely, at least at low gas densities. The leads which supply current to the conductor can be arranged in a way such as to prevent particle losses out of the confinement volume. Modifications and improvements of the magnetic field configuration by the addition of auxiliary coils are discussed.

Injection of ions into the confinement volume is discussed briefly as well as a number of other possible heating mechanisms. A short consideration of the stability problem is also given at the end of the present paper.

1. INTRODUCTION

IN a self-sustained thermonuclear fusion reactor conditions have to be established where the produced energy is sufficiently large to cover all losses as well as a certain amount of useful extracted energy. Radiation losses by bremsstrahlung can be covered if the temperature is kept high enough. At the same time the contact between the reacting gas and surrounding walls should be reduced as far as possible. This will be the case if instabilities of the gas are suppressed and the particle losses minimized. The present paper will deal mainly with particle losses out of a magnetic 'bottle', without taking the stability problem into special consideration.

A lower limit of the confinement time is readily estimated from a comparison between the reaction energy and the energy which is lost when the particles escape from the reaction volume. Especially, for the total DD-reaction with cross-section σ, reaction rate parameter $\langle\sigma v_{DD}\rangle$ m^3/sec, ion density n m^{-3} and released energy W joules per reaction the produced power per unit volume is (POST, 1956; THOMPSON, 1957)

$$P = 1/2n^2\langle\sigma v_{DD}\rangle W \text{ watts/m}^3. \quad (1)$$

A lower limit, $t_{\min}$ sec, of the confinement time at the temperature T°K is now obtained by assuming that only ions are lost, that the entire mass of gas is burnt into ^{4}He (W corresponding to about 21·6 MeV) and that all the produced energy can be used to cover the energy which escapes with the particles to the surroundings:

$$1/2n^2\langle\sigma v_{DD}\rangle W = 3/2nkT \cdot t_{\min}, \quad (2)$$

and

$$t_{\min} = 1{\cdot}20 \times 10^{-11} T/n\langle\sigma v_{DD}\rangle. \quad (3)$$

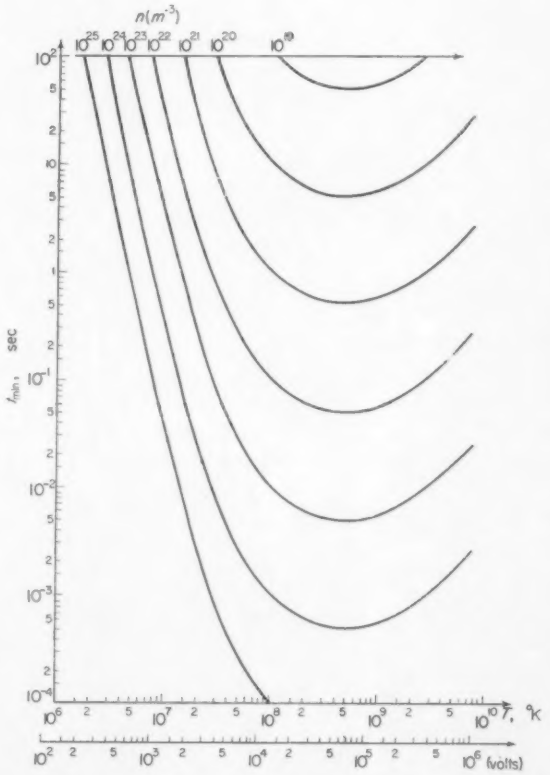

FIG. 1.—A lower limit, $t_{\min}$, of the confinement time which is required for a balance between the energy produced by the DD-reaction and the energy lost with particles escaping from the reaction volume. T is the temperature and n the particle density.

From an empirical formula of the reaction rate given by THOMPSON (1957) the corresponding times are calculated as shown by Fig. 1. The minimum at

about 5×10^{8}°K is caused by the fact that the reaction rate increases only slowly at very high temperatures, whereas the energy required to heat the ions is proportional to T.

It is seen from Fig. 1 that the requirements on the confining properties of a self-sustained thermonuclear reactor are very high. As an example, even a pressure as high as 1000 atm ($\approx 1{\cdot}01 \times 10^{8}$ newtons/m^2) gives a confinement time which has to exceed at least 3×10^{-2} sec when the temperature is of the order of 5×10^{8}°K. For 'magnetic bottles' with field lines leading from the confinement volume out to the surrounding space, such as mirror machines, GARREN *et al.* (1958) point out that scattering losses at the mirror ends present a severe problem for the energy balance. Considering only ion–ion collisions in a uniform, neutral plasma GARREN *et al.* find that temperatures of the order of 3×10^{9}°K have to be established for self-sustained operation of a mirror machine with deuterium.

Even if losses along the field lines are inhibited by using configurations where the lines are closed upon themselves inside the confinement region, particles may be lost through drift motions across the magnetic field. Losses of this type can be eliminated, either by choosing a magnetic field where the drift motions cancel for particles running through the confinement region repeatedly (SPITZER, 1958), or by having drift motions which follow closed paths inside the same region (LEHNERT, 1958a; CHRISTOFILOS, 1958).

Finally, it should be pointed out that the problem of establishing a self-sustained reaction may run into severe difficulties if the diffusion of particles across the magnetic field is accelerated by collective processes, such as 'drain diffusion' (BOHM *et al.*, 1949; LEHNERT, 1958b; HOH and LEHNERT, 1959). If this type of increased diffusion rate is moderate for the conditions of a thermonuclear reactor, however, self-sustained reactions may still be realizable, provided that the confining properties of the magnetic 'bottle' are improved as far as possible.

This paper will deal mainly with the confinement of charged particles in the magnetic field of a *solid* ring-shaped conductor, as suggested earlier by the author (1958a). In terms of ALFVÉN's perturbation theory for the motion of a charged particle (ALFVÉN, 1950) it is easily seen that the drift motions form closed paths around the symmetry axis of the loop. The same result can also be deduced from STÖRMER's equations (STÖRMER, 1955; see also BIERMANN and SCHLÜTER, 1957; LÜST and SCHLÜTER, 1957; FISSER and KIPPENHAHN, 1959).

2. MOTION OF A CHARGED PARTICLE IN STATIC ELECTRIC AND MAGNETIC FIELDS

The non-relativistic Lagrangian of a charged particle of mass m, charge q and velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$, having the vector potential $\mathbf{A}$ and a field from an electrostatic potential ϕ is (GOLDSTEIN, 1957)

$$L = \tfrac{1}{2}mv^2 - q\phi + q\mathbf{v} \,.\, \mathbf{A}. \tag{4}$$

With the components of the generalized co-ordinates, q_i, and generalized momenta, $p_i = \partial L/\partial \dot{q}_i$, the Hamiltonian becomes

$$H = 1/2m(\mathbf{p} - q\mathbf{A})^2 + q\phi. \tag{5}$$

The total time derivative of a quantity Q is

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + [Q, H];$$

$$[Q, H] = \sum_i \left(\frac{\partial Q}{\partial q_i} \cdot \frac{\partial H}{\partial p_i} - \frac{\partial Q}{\partial p_i} \cdot \frac{\partial H}{\partial q_i} \right), \tag{6}$$

where the right hand term is Poisson's bracket.

Any quantity Q which does not depend explicitly on time and whose Poisson bracket with the Hamiltonian vanishes, is a constant of the motion.

For axisymmetric, stationary configurations the magnetic field becomes

$$\mathbf{B} = \left| -\frac{\partial A_\phi}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) \right| \tag{7}$$

in cylindrical co-ordinates (r, φ, z), and the electric field

$$\mathbf{E} = \left| -\frac{\partial \phi}{\partial r}, 0, -\frac{\partial \phi}{\partial z} \right|. \tag{8}$$

equation (6) gives $\dot{p}_\phi = 0$ and

$$p_\phi = mrv_\phi + qrA_\phi = mr_0 v_{\phi_0} + qr_0 A_{\phi_0}, \tag{9}$$

where index ($_0$) denotes values at the starting point of the particle. The result (9) can also be derived from the equation expressing conservation of momentum, as shown, e.g., by COSSLETT (1950), LÜST and SCHLÜTER (1957), and in the theory of the magnetron (BRONWELL and BEAM, 1947). The energy theorem is obtained from equation (6) by putting $Q = H$;

$$1/2m(v^2 - v_0^2) = q(\phi_0 - \phi). \tag{10}$$

Combination of equations (9) and (10) with $v^2 = v_r^2 + v_\phi^2 + v_z^2$ gives

$$v_r^2 + v_z^2 = v_0^2 + \frac{2q}{m}(\phi_0 - \phi) - \frac{1}{r^2}\left[r_0 v_{\phi_0} + \frac{q}{m}(r_0 A_{\phi_0} - rA_\phi) \right]^2 \geqslant 0. \tag{11}$$

If this condition cannot be satisfied a particle starting at a point (r_0, φ_0, z_0) will not reach the point (r, φ, z). As pointed out by LÜST and SCHLÜTER (1957) and which is also seen from equations (7) and (11) a complete particle confinement can be established only by a poloidal magnetic field (B_r, B_z) and not by a toroidal field (B_ϕ).

Starting from the equations of motion of a purely poloidal field COSSLETT (1950) studies the movement in the rz-planes through the axis of symmetry. This may be considered to take place in a potential 'trough' given by the potential

$$U = \phi - \phi_0 + \frac{m}{2qr^2}\left[r_0 v_{\phi_0} + \frac{q}{m}(r_0 A_{\phi_0} - r A_\phi)\right]^2. \quad (12)$$

The same conclusion can also be drawn directly from equation (11). Consider the special case of a strong magnetic field where the radius of gyration of a charged particle is much less than the characteristic dimensions of the magnetic field configuration and where the electrostatic force $q\mathbf{E}$ does not exceed the Lorentz force $q\mathbf{v} \times \mathbf{B}$ considerably. A particle will move close to the 'bottom line' of the trough which consists of the projection $rA_\phi = \text{constant}$ of the corresponding magnetic field line in the rz-planes. Consequently, a magnetic field the poloidal part of which forms closed field lines is able to trap particles which have an energy less than a certain limit.

3. CONFINEMENT BY MEANS OF A CURRENT LOOP

According to the previous section charged particles cannot be trapped in a purely toroidal magnetic field, which is stationary. In terms of the macroscopic equations of an ionized gas SPITZER (1952) and BIERMANN and SCHLÜTER (1957) have drawn the same conclusions for the balance between the pressure and the electromagnetic force in a plasma. However, in a field which has a poloidal part particles can be trapped under certain conditions. In the 'Stellarator' machines this is done by introducing a small perturbation on an essentially toroidal field, e.g., by means of helical windings (KRUSKAL *et. al.*, 1958; SPITZER, 1958). The magnetic coils of the 'Stellarator' are all situated outside the reaction volume and this limits the strength of the poloidal field considerably. An improvement of the confining ability, at least at low gas densities, can be obtained by placing a circular coil 'inside' the reaction volume as shown in Fig. 2. This produces a strong, poloidal magnetic field. A toroidal component may as well be added to generate a screw field. Current is supplied to the coil through a number of leads which are placed in the plane of the coil as suggested in Fig. 2. Alternatively, the leads may be extended radially inwards and a magnetic mirror action will then prevent most of the particles from reaching the lead region. All the conductors in Fig. 2 have to be screened electrostatically to eliminate the influence of electric fields generated by the potential drop in the windings.

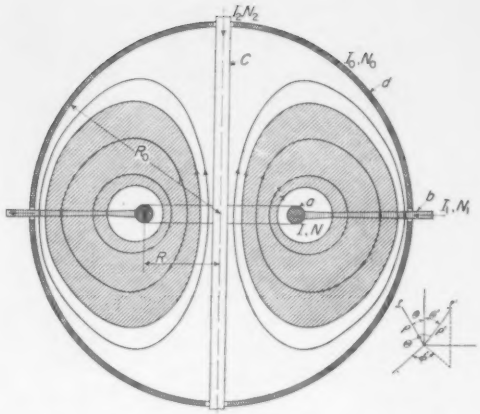

FIG. 2.—Outline of the magnetic field pattern generated by a circular coil a with radius R and a spherical coil d which cancels the field from the circular coil outside the sphere $\rho = R_0$. Current is supplied to the circular coil through a number of double leads b. The shaded area represents the confinement volume; the leads are screened from particle bombardment by their self-fields generated by the currents I_1. A toroidal component may be added to produce a screw field by means of the conductor c traversed by the current I_2.

3.1 *The current leads*

The supply of electric current to the circular coil and its mechanical mounting provides one of the essential difficulties with the arrangement shown in Fig. 2. It has earlier been suggested (LEHNERT, 1958a) that the supply current through the leads may generate a magnetic field which acts as a screen against particle bombardment. This agrees with conclusions drawn by KADOMTSEV and BRAGINSKY (1958) about the motion of a plasma in the neighbourhood of a wire traversed by an electric current. The magnetic field produced by the leads generates additional drift motions, but these cancel for particles running around the current loop repeatedly. BONNEVIER (1958) has pointed out that leads which are arranged in pairs with opposite currents diminish the range of the field arising from the supply currents.

The motion of a charged particle in the magnetic field of a straight line current has recently been considered by HERTWECK (1959). In the present paper a study will be made of the particle motion in the field $\mathbf{b}$ of a double lead, placed in a homogeneous field $\mathbf{B}_0$ as shown in Fig. 3. This forms an idealization of the actual configuration to a plane one with

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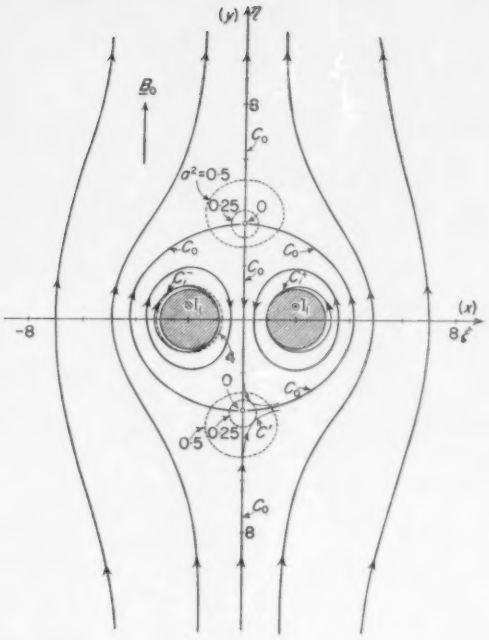

FIG. 3.—Cross-section of a pair of 'force-free' leads with radii r_1 placed at a distance 4 r_1 between their central lines. The ratio $F = b_1/B_0 = 4$ between the self-field b_1 at the surface of a lead and the external magnetic field B_0. The field lines given by C_i^+ and C_i^- graze the surfaces of the leads. C_0 gives the boundary between the field which circulates around the leads and the field which runs to infinity. Broken lines give boundaries inside of which the field has a strength falling below $\alpha^2 B_0$.

infinitely long leads. A co-ordinate system with the z-axis along the leads and the x-axis through their central lines is introduced. Further, introduce the radius r_1 of a lead, the distance 2 d_1 between the centres of the leads, the current I_1 through each lead, the strength of the corresponding field $b_1 = \mu_0 I_1/2\pi r_1$ at the surface of a lead, and the normalized quantities $\xi = x/r_1$, $\eta = y/r_1$, $f = d_1/r_1$ and $F = b_1/B_0$.

With the directions of the electric currents and the main field B_0 given by Fig. 3 the total field becomes

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b} = \left| 0,\ B_0,\ 0 \right| + FB_0 \left| \frac{\eta}{(\xi+f)^2+\eta^2} - \frac{\eta}{(\xi-f)^2+\eta^2},\ -\frac{\xi+f}{(\xi+f)^2+\eta^2} + \frac{\xi-f}{(\xi-f)^2+\eta^2},\ 0 \right|, \quad (13)$$

which can be obtained from the vector potential

$$\mathbf{A} = \left| 0,\ 0,\ B_0 r_1 \left\{ -\xi + 1/2F \log\left[\frac{(\xi+f)^2+\eta^2}{(\xi-f)^2+\eta^2}\right] \right\} + \text{constant} \right|. \quad (14)$$

Very strong magnetic fields are required for the confinement of deuterons at temperatures around 10^{8}°K. Therefore, the conductors producing the field will be subject to large forces. However, the net force on each lead may be cancelled by a proper choice of the mutual distance 2 d_1 and of the ratio F between the field strengths. Then, the mutual force which tends to expand the double lead is balanced by the force arising from the main field $\mathbf{B}_0$ which tends to compress the double lead. The force per unit length acting on the left hand lead of Fig. 3 is in the x-direction and has the value

$$K_x^- = \int\int i_1 B_y \, dx\, dy = -\int\int i_1 \frac{\partial A}{\partial x} \, dx\, dy, \quad (15)$$

where $i_1 = I_1/\pi r_1^2$ and the integration is performed over the cross-section of the lead. After introduction of dimensionless variables equation (15) is easily shown to become

$$K_x^-/I_1 B_0 = 1 - \frac{F}{\pi}\int_0^{\pi/2} \log\left(\frac{1+4f^2+4f\cos\phi}{1+4f^2-4f\cos\phi}\right) \times \cos\phi \, d\phi,$$

and after one partial integration followed by the substitution $t = tg(\phi/2)$ and a partial fraction expansion the result becomes

$$K_x^- = I_1 B_0 (1 - F/2f). \quad (16)$$

The condition for a 'force-free' lead is

$$F = 2f. \quad (17)$$

The force-free position is stable as far as motions in the x-direction are concerned; a value of f less than $F/2$ will give a net expanding force and a value greater than $F/2$ acts in the opposite direction.

Condition (17) is now assumed to be valid throughout the paper and the equation of the field lines, A_z = constant, becomes

$$\frac{(\xi+f)^2+\eta^2}{(\xi-f)^2+\eta^2} = Ce^{\xi/f}, \quad (18)$$

where C is a parameter. The field lines which graze the right hand and left hand leads are determined by

$$C_i^+ = (2f-1)^2 \exp\left[\frac{1}{f} - 1\right]; \quad C_i^- = 1/C_i^+, \quad (19)$$

respectively. $C = C_0 = 1$ corresponds to the field line which follows the η-axis as well as the border between the field which encircles the leads and the field which runs to infinity. According to equation (13) the magnetic field vanishes along the lines

$\xi = 0$, $\eta = \pm\sqrt{3}\,f$. Values of B/B_0 in the vicinity of the zero lines are given by the dotted curves in Fig. 3, which demonstrates the case $f = 2$.

A discussion of the forbidden zones can be carried out along the same lines as in Section 2. From $\dot{p}_z = 0$

$$p_z = mv_z + qA_z = mv_{z_0} + qA_{z_0}, \tag{20}$$

and in combination with equation (10)

$$v_x^2 + v_y^2 = v_0^2 + \frac{2q}{m}(\phi_0 - \phi) - \left[v_{z_0} + \frac{q}{m}(A_{z_0} - A_z)\right]^2. \tag{21}$$

Addition of a field b_z, independent of z, does not alter these results. Inhibition of the particle bombardment of the leads will now be discussed for the most unfavourable situation. This occurs when a particle can move along the field from infinity up to a 'starting point' at the border C_0, when $q(\phi_0 - \phi)$ is positive and when the sign and magnitude of v_{z_0} is such as to make the square bracket term in expression (21) as small as possible. From equations (14) and (18)

$$A_{z_0} - A_z = 1/2 b_1 r_1 \log (C_0/C). \tag{22}$$

A particle starting from C_0 will not reach the leads if

$$\{v_{z_0} \mp (qb_1r_1/m)[\log (F-1) + 1/F - 1/2]\}^2 > v_0^2 + \frac{2q}{m}(\phi_0 - \phi) \equiv u_0^2, \tag{23}$$

where the two signs in the left hand member correspond to C_i^+ and C_i^-, respectively. Condition (23) is fulfilled, even in the most unfavourable situation if the second term in the left hand member has a modulus exceeding $u_0 + v_0$, where u_0 is defined as a positive quantity. Introduce the radius of curvature $a_1 = mv_0/|q|\,b_1$ in the field b_1 of a lead. The 'cut-off' value of a_1/r_1, where a particle grazes a lead, is given by

$$1/2(1 + u_0/v_0)\left(\frac{a_1}{r_1}\right)_c = 1/2[\log (F-1) + 1/F - 1/2] \tag{24}$$

and plotted as a function of F in Fig. 4. The result can be used to calculate the largest allowable radius r_1 for a given external field B_0, potential difference $\phi - \phi_0$, particle energy, and ratio F.

The zero lines at $(0, \pm\sqrt{3}f)$ might at a first consideration be assumed to cause a leakage in the direction along the leads. However, there are reasons for this leakage to be extremely small. Suppose that a particle starts with velocity v_1 at a point P_1 situated

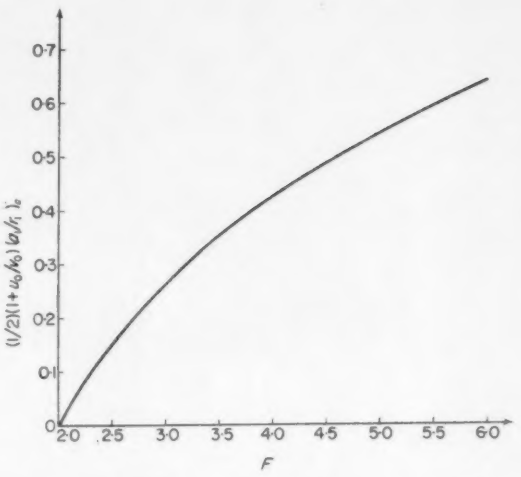

FIG. 4.—Determination of the cut-off value $(a_1/r_1)_c$ as a function of the ratio F for a pair of force-free leads.

at some distance from the leads and from the zero line, where the magnetic field is approximately the undisturbed field $\mathbf{B}_0$. Only particles which are allowed to reach the zero line have to be discussed here. To simplify the treatment the electric field is now assumed to vanish. Suppose also that the magnetic field is chosen strong enough for Alfvén's perturbation theory to be valid up to a point P' where the particle passes a certain surface S' enclosing the zero line (cf the dashed lines in Fig. 3). At the zero line, however, the perturbation theory becomes inapplicable. The surface S' defines a boundary inside of which the magnetic field strength falls below $\alpha^2 B_0$, where $0 < \alpha < 1$. According to the perturbation theory the particle which gyrates around the field lines with the velocity w has a constant equivalent magnetic moment, $mw^2/2B$. In a strong field w is much larger than the drift motion across the field and the velocity component v_z' at P' has a modulus

$$|v_z'| \leq w' = \alpha w_1 \leq \alpha v_1. \tag{25}$$

Now, reconsider equation (21) for the particle which passes P' and reaches the zero line, where the velocity in the xy-plane is given by

$$v_{x_0}^2 + v_{y_0}^2 = v_1^2 - (v_z' + \frac{q}{m} A_z')^2, \tag{26}$$

since $A_{z_0} = \text{constant}$ $\log C_0 = 0$. The second term in the bracket of equation (26) has a modulus

$$\left|\frac{q}{m} A_z'\right| \leq 1/2 v_1 \left(\frac{r_1}{a_{11}}\right) |\log C'|_{\max}, \tag{27}$$

where $a_{11} = mv_1/|q|\,b_1$ and $|\log C'|_{\max}$ is the largest value of $|\log C|$ existing on S'. The conditions in the

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vicinity of the zero point $(0, -\sqrt{3}f)$ are now studied by introducing the variables $\delta = \eta - \sqrt{3}f$ and $\lambda^2 = \xi^2 + \delta^2$. From equation (18) the field lines are easily seen to be equilateral hyperbolas

$$\xi\delta \approx 2(1 - 1/C)f^2/\sqrt{3} \tag{28}$$

and the surfaces $B =$ constant are given by the circles

$$B^2 \approx 3/4B_0{}^2\lambda^2/f^2. \tag{29}$$

The largest value of $|\log C'|$ for a given surface S' with radial distance $\lambda' r_1$ from the zero point is determined by the field line which touches S'. From equation (28) this gives

$$|\log C'|_{\max} \approx \frac{\sqrt{3}}{4}(\lambda'/f)^2. \tag{30}$$

Finally, equation (29) gives a characteristic distance $B/|\nabla B| \approx \lambda' r_1$. The perturbation theory will then be applicable on and outside of S' if

$$a_{11}/r_1 = k'B/|\nabla B|\, r_1 = k'\lambda';\ k' \ll 1. \tag{31}$$

In such a case

$$\left|\frac{q}{m} A_z'\right| \leq \frac{\sqrt{3}}{8} v_1\lambda'/k'f^2. \tag{32}$$

Thus, for given values of v_1, k' and f a magnetic field which is chosen strong enough will make the bracket in equation (26) less than $v_1{}^2$. Then, the velocity in the xy-plane will not vanish at the zero point and the particle will not pass out of the configuration along the zero line. The situation of Fig. 3 with $f = 2$ may serve as an illustration. Suppose that the surface S' is chosen where $\alpha^2 = 0{\cdot}25$. Then, $\lambda' \approx 0{\cdot}58$ and if $k' = 0{\cdot}1$, which gives a good approximation for the perturbation theory, we have $|qA_z'/m| \leq 0{\cdot}31v_1$ and from equation (26) follows $v_{x_0}{}^2 + v_{y_0}{}^2 \geq 0{\cdot}35v_1{}^2$.

The only way in which a particle can escape along the zero lines would be through a collision in their immediate neighbourhood. In addition, the particle should be scattered into a very narrow solid angle to find its way out along a zero line. Finally, it should be pointed out that the mean free path is likely to be very long compared to the dimensions of the leads in an ionized gas at temperatures and densities of thermonuclear interest. Also, the volume of the region nearest to the zero lines, where a leakage due to scattering could take place, is very small compared with the entire reaction volume. Consequently, it may be justified to assume that the zero lines do not introduce any dangerous particle leakage, at least at moderate densities and strong magnetic fields.

In the plane configuration of Fig. 3 discussed so far the influence of the curvature of the main field has been neglected. However, if the distance between the leads at any cross-section is chosen properly to give a force-free situation such as that shown in Fig. 3 the conclusions drawn from the plane case will not be far from the situation of Fig. 2. This is made plausible by the fact that the surfaces $B =$ constant very nearly coincide with the field lines in the immediate neighbourhood of the lead surface (cf the dotted line for $\alpha^2 = 4$ in Fig. 3). In terms of the perturbation theory the particle motion can be divided into one component along the field and a drift velocity, $mw^2\mathbf{B} \times (\nabla B)/2qB^3$, contained in the surfaces $B =$ constant. This implies that the particle moves very nearly parallel with the lead surface if it may reach its immediate neighbourhood. In addition, the field in this region is about F times stronger than the undisturbed field $\mathbf{B}_0$ and this will prevent a large fraction of particles from reaching the region close to the leads. Consequently, the leads should have a very strong tendency to reflect particles if the magnetic field is made strong enough.

From the point of view of the perturbation theory it has earlier been shown that the drift motions introduced by the main, poloidal field in Fig. 2 circulate around the axis of symmetry and cause no particle leakage to the surroundings. If a toroidal component is added to the main field the rigorous theory shows that the particle confinement is still complete, as long as a poloidal component is present. The plane through the centres of the double leads has to be somewhat inclined to the plane of the circular coil to preserve the force-free situation in such a screw field. Additional drift motions will be introduced by the leads, essentially in the direction of their axis, and will depend upon the sign of the supply current. However, the drift motion encircling the symmetry axis of the main coil and the screw field will force the particles to circulate around the axis of symmetry. Statistically they will spend as much time in the region of a lead which causes a drift in one direction as in the region of a lead causing an opposite drift motion. A similar cancellation will govern particles which pass the channels around the zero lines. Thus, the leads are likely to cause no severe particle leakage out of the confinement region indicated in Fig. 2. Owing to the curvature of the configuration in Fig. 2 a weak field $b_z \ll B_0$ arises along the zero lines. This does not alter the conclusions obtained so far.

3.2 *The main field*

Part of the currents producing a field which is closed inside the intended confinement volume can

be situated in the surrounding space. In Fig. 2 a spherical coil with radius R_0 has been placed around the circular coil with radius R. This is advantageous in several respects. The current distribution along the sphere can be chosen such as to cancel the field of the circular coil in the external region $\rho > R_0$. The field used to trap particles will then be 'compressed' into the space $\rho < R_0$ and the volume which is available for the confinement is increased. It also reduces the gradient of the main field along the leads. A considerable part of the currents used for the field generation is removed to regions where large conductor cross-sections can be used and the ohmic losses kept low. Finally, if the coils are situated inside a vessel made of metal changes in the main field will not induce currents in the vessel walls. Otherwise, these currents would introduce disturbing fields out of phase with the main field which violate the force-free condition for the leads. The total current I circulating in the loop and the total currents I_0 and $I_{\frac{1}{2}}$ in the spherical coil d and in the conductor c of Fig. 2 should be kept in phase when they vary in time, which can be obtained by connecting all the coils in series. If some sub-sections of the spherical coil have to be connected in parallel their currents will still be in phase when the current distribution is chosen such as to balance the external field for $\rho > R_0$.

As an alternative to the confinement region of Fig. 2 the region nearest to the circular coil could be used, the coil being enclosed in a large vessel shaped as a torus.

The current distribution on the spherical surface $\rho = R_0$ which cancels the external field is deduced in the following way. A spherical system of co-ordinates (ρ, θ, ϕ) is introduced. A distribution of currents with density $\mathbf{i}(\mathbf{r}') = i(\rho', \theta') \cdot \hat{\boldsymbol{\Phi}}$ is assumed on and inside the sphere with radius R_0 (see also MAXWELL, 1873). The vector potential at a field point $\mathbf{r}$ becomes

$$\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi) \int\int\int \frac{\mathbf{i}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \rho'^2 \sin\theta' \, d\rho' \, d\theta' \, d\phi'. \tag{33}$$

For the denominator the expansions

$$|\mathbf{r} - \mathbf{r}'| = \frac{1}{\rho} \sum_{l=0}^{\infty} \left(\frac{\rho'}{\rho}\right)^l P_l(\cos\theta) \qquad \rho' < \rho \tag{34}$$

and

$$|\mathbf{r} - \mathbf{r}'| = \frac{1}{\rho'} \sum_{l=0}^{\infty} \left(\frac{\rho}{\rho'}\right)^l P_l(\cos\theta) \qquad \rho' > \rho \tag{35}$$

are used. Further, the addition formula for the Legendre polynomials is

$$P_l(\cos\Theta) = P_l(\cos\theta') \, P_l(\cos\theta) + 2 \sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} \times P_l^m(\cos\theta') \, P_l^m(\cos\theta) \cos m(\phi - \phi'). \tag{36}$$

Substituting expressions (34), (35) and (36) into equation (33) and carrying out the integration $\mathbf{A}$ is seen to have a component in the ϕ-direction only.

Especially for a spherical coil with surface current density $i(t)$, where $t = \cos\theta'$ the vector potential becomes

$$A_\phi(\rho, \theta) = 1/2 \, \mu_0 \sum_{l=1}^{\infty} \frac{i_l}{l(l+1)} \rho^{-l-1} P_l^1(t); \quad \rho > R_0 \tag{37}$$

$$i_l = R_0^{l+2} \int_{-1}^{1} i(t) P_l^1(t) \, dt; \qquad \rho > R_0 \tag{38}$$

and

$$A_\phi(\rho, \theta) = 1/2 \, \mu_0 \sum_{l=1}^{\infty} \frac{i_l}{l(l+1)} \rho^l P_l^1(t); \qquad \rho < R_0 \tag{39}$$

$$i_l = R_0^{-l+1} \int_{-1}^{1} i(t) P_l^1(t) dt; \qquad \rho < R_0. \tag{40}$$

The functions $P_l^1(t)$ form a complete orthogonal set and $i(t)$ can be expanded according to

$$i(t) = \sum_{l=1}^{\infty} a_l P_l^1(t); \; a_l = \frac{2l+1}{2l(l+1)} \int_{-1}^{1} i(t) P_l^1(t) \, dt. \tag{41}$$

For a current loop of radius R and total current I the corresponding expressions for the vector potential are equation (37) for $\rho > R$ and equation (39) for $\rho < R$, where

$$i_l = R^{l+1} I P_l^1(0); \qquad \rho > R \tag{42}$$

$$i_l = R^{-1} I P_l^1(0); \qquad \rho < R, \tag{43}$$

and $P_l^1(0)$ can be deduced from relations given by JAHNKE and EMDE (1938). The current distribution $i(t)$ on a sphere which cancels the field of the current loop in the external region $\rho > R_0$ is now obtained from $-i_l$ given by equation (41) which is substituted into equation (38). The resulting expression is combined with equation (41) to give

$$i(t) \equiv i(\cos\theta) = \frac{I}{R_0} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(4n+3)(2n)!}{2^{2n+2} n! \, (n+1)!} \left(\frac{R}{R_0}\right)^{2n+2} P_{2n+1}^1(\cos\theta). \tag{44}*$$

This current distribution produces an additional

* The author is indebted to Dr. B. NAGEL for the deduction of equation (44).

magnetic field in the internal region $\rho < R_0$ which is given by the vector potential

$$A_\varphi^{(0)}(\rho, \theta) = \mu_0 I \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2n)!}{2^{2n+2}\, n!\,(n+1)!} \cdot \left(\frac{R}{R_0}\right)^{2n+2} \left(\frac{\rho}{R_0}\right)^{2n+1} P_{2n+1}^1(\cos\theta), \quad (45)$$

as deduced from equations (40), (41) and (39). The first term in equations (45) generates a homogeneous magnetic field along the axis of symmetry in a direction opposite to the dipole moment of the current loop.

4. HEATING MECHANISMS

Practically any of the heating mechanisms suggested earlier for obtaining thermonuclear reactions in a plasma can be used in combination with the magnetic 'bottle' described in this paper. Only a few possibilities will be mentioned briefly here.

One possibility consists of the injection of fast ions from an ion source. D^+ ions could be accumulated by dissociation of injected D_2^+ molecular ions by means of collisions (cf. BELL *et al.*, 1958; LUCE, 1958; ARTSIMOVICH, 1958). Alternatively, the ions may be injected in a screw field as shown in Fig. 5. The pitch of the field lines and the magnitude of the drift velocity encircling the axis can be determined such as to prevent the particles from hitting the back side of the ion source. A change in the magnetic field will induce an electric field $\mathbf{E}$ and a corresponding transverse velocity $\mathbf{u}_\perp = \mathbf{E} \times \mathbf{B}/B^2$. According to SPITZER (1956a) the particle will move on a surface where the magnetic flux is conserved. When a complete passage around the axis of symmetry has been performed the magnetic field strength is assumed to have decreased slowly to allow the particle to pass 'inside' the source. A slow increase in the field strength might as well have been applied in which case the particle will pass 'outside' the source.

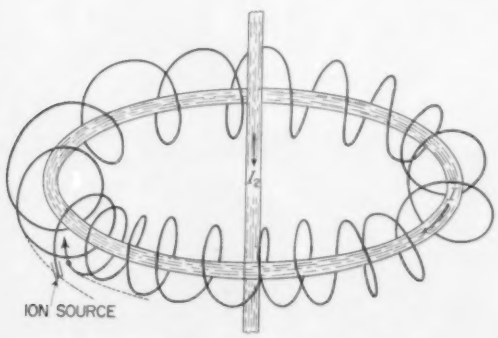

FIG. 5—Injection of ions from an ion source into a screw field produced by a ring current I and a current I_2 generating a toroidal field component. In the figure the field strength is assumed to decrease slowly to allow the ions to pass inside the source after one passage around the axis of symmetry. In the actual case a particle will gyrate around the path shown by the figure.

A difficulty with the injection of an ion beam is introduced by the electrostatic forces produced by its charge, which limit the particle density. This can be overcome by injecting electrons simultaneously. However, in such a case it should be observed that unevenly distributed space charges may occur in the early stages of the injection period which may produce additional drift motions and particle losses. Consequently, the reaction volume may have to be filled in advance with an evenly distributed electron cloud to permit the injection of ions.

Another possibility is provided by the mechanism applied in the 'Homopolar' and 'Ixion' experiments (BAKER, 1956; BOYER, 1957). An electrostatic voltage may be applied between the circular coil in Fig. 2 and the central conductor c which is connected with an axisymmetric electrode following the inner surface of the spherical coil. At least parts of the plasma would then be set into rotation permitting a strong electric field to penetrate its boundary layers. The crossed electric and magnetic fields produce trochoidal particle paths around the axis of symmetry where the gyrating particles collide at high relative velocities, provided that the electric field is strong enough.

Finally, any mechanism can be used which is based upon the heating of a plasma by an oscillating electromagnetic field, eventually superimposed on the basic confining field (cf. SPITZER and WITTEN, 1953; BERGER *et al.*, 1958). BERGER and co-workers point out that these mechanisms will add energy to the plasma, provided that the confinement is perfect.

5. CONCLUDING REMARKS ON THE STABILITY PROBLEM

Confinement of charged particles in the present field configuration is expected to be very effective at low gas densities. At higher densities, however, the stability problem also has to be taken into account.

Among the possible hydrodynamic instabilities are those of 'kink' and 'sausage' type known from pinch experiments. A straight counterpart to the present configuration is a hollow pinch enclosing a conductor which is traversed by an electric current. In an experiment with such a 'tubular pinch' BAKER and ANDERSON (1956) found that it was stable for much longer periods than a simple pinch filament. This also agrees with conclusions drawn by ANDERSON *et al.* (1958). Qualitatively this may be understood from the fact that part of the magnetic field is generated by currents 'anchored' in the solid conductor. How

much the curved geometry of a current loop may modify these results cannot be judged at this stage.

Under certain circumstances 'exchange instabilities' may arise where, so to speak, the plasma and the magnetic field interchange positions. According to SPITZER (1956b) these instabilities can be inhibited by using a twisted magnetic field with "shear", i.e., where magnetic field lines in different layers run with different pitch angles. The present configuration is suitable for the generation of such fields, even with a higher degree of 'shear' than that which can be obtained in the 'Stellarator'.

When the confinement region shown in Fig. 2 is uniformly filled with a quasi-neutral plasma there are no drift motions which may cause charge separation and additional drifts which bring particles out to the walls. In addition, the screw field and the drift velocities produced by the inhomogeneity of the magnetic field give a rotational transform, similar to that in the 'Stellarator', which counteracts the generation of local space charges.

Among the instabilities of non-hydrodynamic character the production of "run-away electrons" (DREICER, 1958) may cause considerable energy and particle losses during the heating process, provided that the accelerating electric field has a component in the direction of the magnetic field. However, for the heating mechanisms suggested in section 4 an electric field component in this direction can be avoided.

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EQUILIBRIUM CONFIGURATIONS OF A TOROIDAL PLASMA

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Abstract—Equilibrium configurations of a toroidal plasma have been obtained, characterized by parameters for the gas pressure and the toroidal component of the magnetic field and by the aspect ratio of the torus. These configurations are symmetrical about the axis of the torus and fall naturally into two classes, corresponding to cases where the toroidal component of the magnetic field either is in the same direction over a cross-section of the torus or changes direction in the centre of the cross-section.

1. INTRODUCTION

IT is usually assumed as a first approximation that the behaviour of a toroidal plasma can be discussed by considering a cylindrical plasma of finite length and imposing suitable periodic boundary conditions on the ends of the cylinder. However, SPITZER (1958) has shown that a plasma cannot be contained in static equilibrium in a torus by a magnetic field which has only a toroidal component, while such an equilibrium is possible in a cylinder with only an axial magnetic field.

We give an account of methods by which we have obtained static equilibrium configurations of a toroidal plasma. Several physical assumptions are made in obtaining these results. We assume that the plasma can be considered to be an ideal inviscid fluid with infinite electrical conductivity, so that the theory of magnetohydrodynamics can be applied in its simplest form. We assume that the plasma fills the whole tube and consider only those equilibria which are symmetric about the axis of the torus. The wall of the tube is also assumed to be an ideal conductor, so that the only boundary condition to be applied is that the normal component of the magnetic field at the wall of the tube should vanish.

With this model for the plasma, we have obtained a wide variety of equilibrium configurations, characterized by a set of three parameters, the gas pressure parameter a, the magnetic field parameter b, and the reciprocal of the aspect ratio ε.

In Section 2, we show that the pressure and magnetic field can be expressed in terms of a stream function ψ which is essentially the toroidal component of the vector potential. In Section 3, we consider a restricted class of equilibria for which the problem is tractable and which is still fairly general and non-trivial. Two methods of solution are given. The first method involves an integral equation which expresses the boundary condition on the magnetic field, while the second adopts an expansion procedure about the corresponding cylindrical configuration. The results are discussed in Section 4 and a typical configuration is given in Figs. 2, 3 and 4.

2. THE STREAM FUNCTION

We consider only static equilibrium configurations in which there is no toroidal variation. It is then convenient to use cylindrical co-ordinates (ρ, φ, ζ) as in Fig. 1. Thus, we have $p = p(\rho, \zeta)$ and $\boldsymbol{B} = \boldsymbol{B}(\rho, \zeta)$, where p and $\boldsymbol{B}$ are the pressure and magnetic field, respectively.

We shall neglect dissipative effects, assuming that the plasma is an ideal, inviscid fluid with infinite electrical conductivity. Neglecting displacement currents, we obtain the following set of magnetohydrodynamic equations to be satisfied by the equilibrium configuration, in which the plasma is taken to be at rest.

$$\frac{1}{c}\boldsymbol{j} \times \boldsymbol{B} = \operatorname{grad} p \tag{2.1}$$

$$\operatorname{curl} \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} \tag{2.2}$$

$$\operatorname{div} \boldsymbol{B} = 0 \tag{2.3}$$

$$\boldsymbol{E} = 0 \tag{2.4}$$

$$\boldsymbol{v} = 0 \tag{2.5}$$

Here, $\boldsymbol{j}$ is the current density, $\boldsymbol{E}$ the electric field and $\boldsymbol{v}$ the fluid velocity.

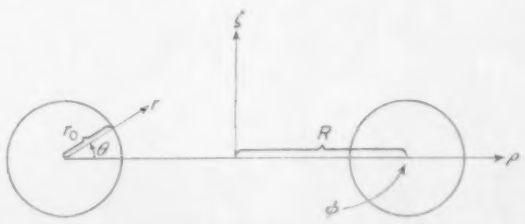

FIG. 1.—Cross-section of torus showing the co-ordinate systems (ρ, φ, ζ) and (r, θ).

Combining equations (2.1) and (2.2), we obtain

$$\text{curl } \boldsymbol{B} \times \boldsymbol{B} = 4\pi \text{ grad } p. \tag{2.6}$$

Equations (2.3) and (2.6) have to be satisfied by the pressure and magnetic field. GRAD and RUBIN (1958) have shown that we can introduce a stream function ψ such that

$$B_\rho = \frac{B_0 R^2}{\rho} \frac{\partial \psi}{\partial \zeta}, \qquad B_\zeta = -\frac{B_0 R^2}{\rho} \frac{\partial \psi}{\partial \rho}. \tag{2.7}$$

B_0 is a constant magnetic field and R is the major radius of the torus, as in Fig. 1. A similar method has been followed by LÜST and SCHLÜTER (1957).

The stream function is in fact simply related to the toroidal component of the vector potential $\boldsymbol{A}$. Thus,

$$A_\varphi = -\frac{B_0 R^2}{\rho} \psi. \tag{2.8}$$

It can be shown, using equation (2.6), that p and ρB_φ are functions of ψ alone. We then introduce arbitrary functions $f(\psi)$ and $g(\psi)$ such that

$$p = \frac{B_0^2}{4\pi} g(\psi), \tag{2.9}$$

$$B_\varphi = \frac{R B_0}{\rho} f(\psi). \tag{2.10}$$

Then, by considering either the ρ-component or the ζ-component of equation (2.6), we find that ψ must satisfy the equation

$$x \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \psi}{\partial x}\right) + \frac{\partial^2 \psi}{\partial y^2} + ff' + x^2 g' = 0 \tag{2.11}$$

where $x = \rho/R$ and $y = \zeta/R$ are non-dimensional variables and a prime denotes differentiation with respect to ψ.

We assume that the plasma fills the whole tube, though for containment we require the pressure to be higher at the centre of the torus cross-section than at the wall. Assuming that the wall is an ideal conductor, we have to impose only one boundary condition, that the normal component B_n of the magnetic field at the wall of the torus should vanish. It is easily seen, on introducing polar co-ordinates r, θ, as in Fig. 1, that the vanishing of B_n is equivalent to the vanishing of $\partial\psi/\partial\theta$.

The two co-ordinate systems are simply related. Thus,

$$\rho = R + r\cos\theta, \qquad \zeta = r\sin\theta. \tag{2.12}$$

In non-dimensional form, we have

$$x = 1 + \varepsilon\xi\cos\theta, \qquad y = \varepsilon\xi\sin\theta. \tag{2.13}$$

$\varepsilon = r_0/R$ and $\xi = r/r_0$, where r_0 is the cross-sectional radius. The torus is defined by ε, which is the reciprocal of the aspect ratio.

3. THE LINEAR ASSUMPTION

Equation (2.11) is in general a non-linear partial differential equation for ψ. It is possible to choose simple forms for $f(\psi)$ and $g(\psi)$ which linearize the equation for ψ, and which still lead to non-trivial equilibrium configurations. We assume

$$f = b\psi, \qquad g = \tfrac{1}{2}a\psi^2. \tag{3.1}$$

In (3.1) we can add a constant to g. a and b are arbitrary parameters. Equation (2.11) then becomes

$$x \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \psi}{\partial x}\right) + \frac{\partial^2 \psi}{\partial y^2} + (ax^2 + b^2)\psi = 0. \tag{3.2}$$

We can now express the pressure and magnetic field in terms of $\psi(x, y)$ and its partial derivatives with respect to x and y.

$$p = \frac{B_0^2}{8\pi} a\psi^2, \qquad B_\rho = \frac{B_0}{x}\frac{\partial \psi}{\partial y}, \qquad B_\varphi = \frac{bB_0}{x}\psi, \qquad B_\zeta = -\frac{B_0}{x}\frac{\partial \psi}{\partial x}. \tag{3.3}$$

(a) *Integral equation method*

Equation (3.2) is separable and one solution is

$$\psi = x\cos ly\, L(kx;\ a/k^4), \tag{3.4}$$

where $L(kx;\ a/k^4)$ satisfies the equation

$$x^2 \frac{d^2 L}{dx^2} + x\frac{dL}{dx} + (ax^4 + k^2x^2 - 1)L = 0 \tag{3.5}$$

and $k^2 + l^2 = b^2$.

A more general solution can be constructed by superposing a continuum of solutions of the form (3.4). Thus

$$\psi(x, y) = x\int_0^\pi d\chi A(\chi)\cos(by\sin\chi/2)L(bx\cos\chi/2;\ \delta). \tag{3.6}$$

In (3.6) we have introduced χ through the relations

$$k = b\cos\chi/2, \qquad l = b\sin\chi/2, \tag{3.7}$$

and $\delta = a/(b\cos\chi/2)^4$.

Thus, we restrict ourselves to real values k and l. Another restriction which we shall make is to consider only the regular solution of equation (3.5), even though the singular point of the equation, $x = 0$, lies outside the domain of the torus. We have not been able to prove that these restrictions will permit the boundary condition on ψ to be satisfied. We have also considered only $\cos ly$ in (3.4) in order that B_φ, through ψ, should be symmetric in the plane of the torus, $\zeta = 0$.

So far, $\mathsf{A}(\chi)$, in expression (3.6), is an arbitrary function of χ. We apply the boundary condition $\psi =$ constant on the wall of the torus. Arbitrarily choosing the constant to be unity (we may have any other number except zero, which is a special case requiring ψ to have a zero on the boundary) we see that $\mathsf{A}(\chi)$ must satisfy the integral equation

$$1 = \int_0^\pi d\chi \mathsf{A}(\chi)(1+\varepsilon\cos\theta)\cos(b\varepsilon\sin\theta\sin\chi/2)\, L[(1+\varepsilon\cos\theta)b\cos\chi/2;\ \delta]. \tag{3.8}$$

This follows from the fact that, on the boundary, $\xi = 1$ and so $x = 1 + \varepsilon\cos\theta$, $y = \varepsilon\sin\theta$.

If, in forming (3.8), we choose the constant to be zero, we have an eigenvalue problem, for the resulting equation has a non-trivial solution $\mathsf{A}(\chi) \not\equiv 0$ only for particular values of the parameters a, b and ε.

It can be shown that the regular solution of equation (3.5) can be expanded as a power series in a. We give here the first four terms of the expansion. Let

$$L(kx;\ a/k^4) = \sum_{n=0}^{\infty}\left(\frac{a}{k^4}\right)^n L_n(kx). \tag{3.9}$$

Then, defining $t = kx$, we find:

$$\begin{aligned}
L_0(t) &= J_1(t),\\
L_1(t) &= -\frac{t^3}{6}J_2(t),\\
L_2(t) &= \frac{t^6}{72}J_3(t) - \frac{t^5}{90}J_4(t),\\
L_3(t) &= -\frac{t^9}{1344}J_4(t) + \frac{143}{90720}t^8 J_5(t)\\
&\quad -\left(\frac{t^7}{945} - \frac{t^9}{36288}\right)J_6(t).
\end{aligned} \tag{3.10}$$

In (3.10), $J_n(t)$ is a Bessel function of the first kind.

(b) *Expansion about cylindrical configuration*

It is possible to obtain approximate toroidal configurations by developing an expansion in powers of ε about the corresponding cylindrical configuration. Equation (3.2) can be expressed in the form

$$\nabla^2\psi - \frac{1}{x}\frac{\partial\psi}{\partial x} + (ax^2 + b^2)\psi = 0, \tag{3.11}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

We now transform equation (3.11) to the polar co-ordinates r,θ. Using the non-dimensional form ξ,θ and expressing $\frac{1}{x}\frac{\partial\psi}{\partial x}$ as grad $\ln x$. grad ψ, we immediately obtain the equation

$$\begin{aligned}
\nabla^2\psi &- \frac{\partial}{\partial\xi}\ln(1+\varepsilon\xi\cos\theta)\frac{\partial\psi}{\partial\xi}\\
&- \frac{1}{\xi^2}\frac{\partial}{\partial\theta}\ln(1+\varepsilon\xi\cos\theta)\frac{\partial\psi}{\partial\theta}\\
&+ \varepsilon^2[a(1+\varepsilon\xi\cos\theta)^2 + b^2]\psi = 0,
\end{aligned} \tag{3.12}$$

where ∇^2 is now $\frac{1}{\xi}\frac{\partial}{\partial\zeta}\xi\frac{\partial}{\partial\xi} + \frac{1}{\xi^2}\frac{\partial^2}{\partial\theta^2}$.

Now, let

$$\psi = \psi_0(\xi) + \varepsilon\psi_1(\xi,\theta) + \varepsilon^2\psi_2(\xi,\theta) + \dots \tag{3.13}$$

and $\alpha = a\varepsilon^2$, $\beta = b\varepsilon$, $\kappa^2 = \alpha + \beta^2$.

We then obtain a set of equations for $\psi_0, \psi_1, \psi_2, \dots$,

$$\begin{aligned}
&(\nabla^2 + \kappa^2)\psi_0 = 0,\\
&(\nabla^2 + \kappa^2)\psi_1 = \cos\theta\left(\frac{d\psi_0}{d\xi} - 2\alpha\xi\psi_0\right),\\
&(\nabla^2 + \kappa^2)\psi_2 = -\xi\cos^2\theta\left(\frac{d\psi_0}{\alpha\xi} + \alpha\xi\psi_0\right)\\
&\quad + \cos\theta\frac{\partial\psi_1}{\partial\xi} - \frac{\sin\theta}{\xi}\frac{\partial\psi_1}{\partial\theta} - 2\alpha\xi\cos\theta\psi_1,\\
&\quad \dots\dots
\end{aligned} \tag{3.14}$$

α, β and κ have been introduced explicitly since these quantities must be kept constant as ε varies, in order to reproduce corresponding configurations for different values of the aspect ratio, and in particular for the cylindrical case, $\varepsilon = 0$.

It can be shown that

$$\begin{aligned}
\psi_0 &= J_0(\kappa\xi),\\
\psi_1 &= \tfrac{1}{2}\cos\theta[\xi J_0(\kappa\xi) - \frac{\alpha}{\kappa}\xi^2 J_1(\kappa\xi) + AJ_1(\kappa\xi)],\\
\psi_2 &= (B\xi^4 + D\xi^2)J_0(\kappa\xi) + (C\xi^3 + E\xi)J_1(\kappa\xi)\\
&\quad + \cos 2\theta[(L\xi^4 + N\xi^2)J_0(\kappa\xi)\\
&\quad + (M\xi^3 + P\xi)J_1(\kappa\xi) + QJ_2(\kappa\xi)],
\end{aligned} \tag{3.15}$$

where

$$
\begin{aligned}
A &= -\frac{J_0(\kappa)}{J_1(\kappa)} + \frac{\alpha}{\kappa}, \\
B &= -\frac{1}{16}\frac{\alpha^2}{\kappa^2}, \\
C &= -\frac{5}{24}\frac{\alpha}{\kappa} + \frac{1}{6}\frac{\alpha^2}{\kappa^3}, \\
D &= -\frac{1}{16} - \frac{1}{12}\frac{\alpha}{\kappa^2} + \frac{1}{6}\frac{\alpha^2}{\kappa^4} + \frac{1}{8}\frac{\alpha}{\kappa}A, \\
E &= \frac{3}{8}\frac{1}{\kappa} + \frac{1}{6}\frac{\alpha}{\kappa^3} - \frac{1}{3}\frac{\alpha^2}{\kappa^5} + \frac{1}{8}A\left(1 - 2\frac{\alpha}{\kappa^2}\right), \\
L &= -\frac{1}{16}\frac{\alpha^2}{\kappa^2}, \\
M &= -\frac{5}{24}\frac{\alpha}{\kappa} + \frac{1}{8}\frac{\alpha^2}{\kappa^3}, \\
N &= -\frac{1}{16} + \frac{1}{8}\frac{\alpha}{\kappa}A, \\
P &= \frac{1}{8}A, \\
Q &= -[(L+N)J_0(\kappa) \\
&\qquad + (M+P)J_1(\kappa)]/J_2(\kappa).
\end{aligned}
\tag{3.16}
$$

A and Q are fixed by the boundary condition $\psi =$ constant at $\xi = 1$, which is obtained by setting the coefficients of $\cos\theta$ and $\cos 2\theta$ equal to zero at $\xi = 1$.

The expansion procedure needs modification when $J_1(\kappa)$ or $J_2(\kappa)$ are zero. If we include higher order terms in the expansion, it is seen that the expansion contains infinite terms when $J_n(\kappa)$ is zero, $n > 0$. We can of course renormalize the solution by multiplying ψ by $J_n(\kappa)$ and then letting κ tend to the zero of J_n. With this renormalization, $\psi_0, \psi_1, \ldots, \psi_{n-1}$ are all zero and a multi-polar cylindrical field is then obtained. However, we shall not be concerned with these cases here.

In terms of the polar co-ordinates used in (b) above, the magnetic field has components $(B_r, B_\theta, B_\varphi)$. This notation may be compared with the notation normally adopted in cylindrical geometry. In the latter case, we usually have $\boldsymbol{B} = (B_r, B_\theta, B_z)$. Thus, B_r and B_θ have the same meaning, but for B_z we have B_φ. We can express B_r and B_θ in terms of ψ. Thus,

$$
\begin{aligned}
B_r &= \frac{B_0}{x\varepsilon\xi}\frac{\partial\psi}{\partial\theta} \\
B_\theta &= -\frac{B_0}{x\varepsilon}\frac{\partial\psi}{\partial\xi},
\end{aligned}
\tag{3.17}
$$

where ψ has been expressed in terms of ξ and θ.

4. DISCUSSION OF RESULTS

Equilibrium configurations have been obtained, using the Ferranti Mercury computer, for a range of values of the parameters a, b and ε. a/b^2 is essentially the ratio of the gas pressure to the magnetic pressure at the centre of a cross-section of the torus. It is, in fact, the ratio of the gas pressure to the magnetic pressure associated with the toroidal component of the magnetic field but the two components in the cross-sectional plane are typically small at the centre.

The results fall into two classes, depending on whether the toroidal magnetic field either is in the same direction over a cross-section of the torus, or has its direction reversed near the wall of the torus. To proceed continuously from one class to the other, we have to pass through a zero of the stream function ψ on the boundary of the torus. This appears as a singularity of the integral equation given in Section 3(a), which expresses the condition that ψ should be unity on the wall of the torus. No difficulty arises in the corresponding case using the expansion procedure of Section 3(b).

We show graphically a typical case in which $a = 17$, $b = 8$ and $\varepsilon = \frac{1}{4}$. We compare results obtained using the two methods outlined in Section 3. The results of the expansion method are shown by the solid curves, while those of the integral method are shown by crosses and by a broken curve where a deviation from the expansion method occurs. Fig. 2 shows the stream function ψ as a function of $\xi = r/r_0$ for three values of θ, $\theta = 0$, $\pi/2$ and π. Fig. 3 shows the magnetic field components B_r, B_θ and B_φ and Fig. 4 shows the variation in pressure, apart from an arbitrary constant. It can be seen that agreement between the two methods is very good. It should be pointed out that, for the sake of comparison, the results using the integral method have been normalized so that ψ is unity at the centre of the cross-section rather than at the boundary.

It is noteworthy that the magnetic fields obtained show a certain similarity to the magnetic fields observed in discharges in ZETA, although there is no obvious reason why this should be so. More detailed comparison should be possible when more experimental results are forthcoming.

We can extend the method of Section 3(b) to obtain results when the arbitrary functions $f(\psi)$ and $g(\psi)$ are assumed to have the forms $f = b\psi + c$ and $g = a\psi^2 + d\psi$. With the extra parameters c and d at our disposal, it should be possible to obtain a much wider class of equilibria. For example, when c and d vanish, a change of sign of B_φ is necessarily accompanied by a minimum of p.

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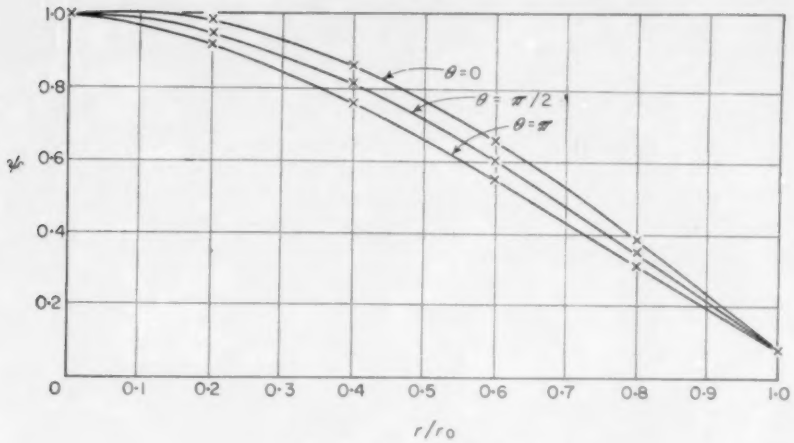

FIG. 2.—Stream function ψ.

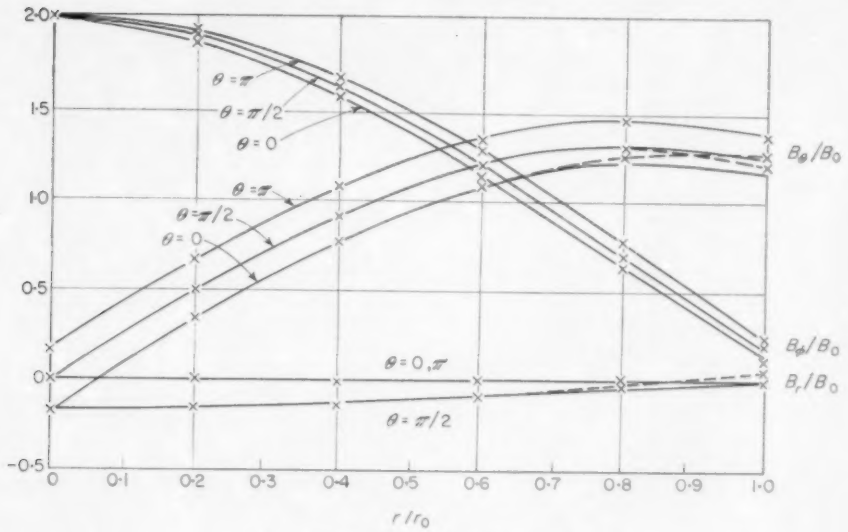

FIG. 3.—Magnetic field $\boldsymbol{B} = (B_r, B\theta, B_\varphi)$.

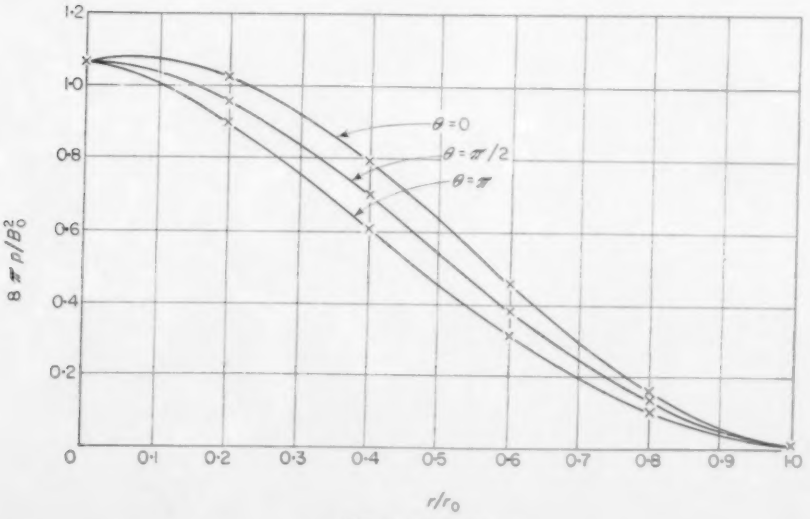

FIG. 4.—Pressure p.

It should be emphasized that we have discussed only a small sub-class of possible equilibrium configurations. A simpler set of configurations can be obtained, using the methods given above, by choosing f to be constant. However, B_φ is then proportional to $1/\rho$, which is too restrictive a condition. It is unlikely that such a choice would lead to interesting equilibrium configurations of a toroidal gas discharge.

As to the stability of the equilibria obtained, little can be said at present. If we apply the stability criterion obtained by SUYDAM (1958), we find that the cylindrical configuration to which the toroidal configuration corresponds is always unstable. It is not at all clear whether the instabilities will remain or be removed in the transition to toroidal geometry.

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AN EXPERIMENTAL AND THEORETICAL INVESTIGATION OF THE MAGNETIC FIELD PRODUCED BY RIDGED POLE-PIECES

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Abstract—An experimental investigation is described of the magnetic field distribution arising from straight parallel ridges of rectangular cross-section on the poles of a magnet. The 'field flutter' and 'mean gap' are defined and summarized graphically for an extensive range of ridge configurations, and for mean fields up to 20 kG. By comparison with the fields produced by spiral ridge configurations it is shown that the results are directly applicable to the design of the spiral ridge pole-pieces required by certain types of particle accelerator.

The experimental investigation is supplemented by a number of theoretical calculations which enable the results to be extrapolated outside the range covered by the measurements. Brief discussions are included of the harmonic content of the field waveforms and of the use of ridge cross-sections other than rectangular.

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1. INTRODUCTION

RECENT trends in the design of orbital particle accelerators have included extensive discussion of the fixed field alternating gradient (FFAG) family of accelerators (SYMON *et al.*, 1956; DUNN *et al.*, 1956). The principle will not be discussed here, but it involves as its basic feature the creation of an aximuthally periodic magnetic field superimposed on the main guide field. It has been shown that under certain conditions this so-called 'flutter field' has the effect of providing vertical focusing for the particles.

The required field variation can be produced by a system of ridges on the magnet poles. In an accelerator of the cyclotron type these ridges may be spiral in shape and the pole-piece could have an appearance similar to that shown in plan view in Fig. 1.

At the high fields which would normally be used in such accelerators (>10 kG), saturation effects in the steel make it difficult to calculate theoretically the field distribution which would be produced by a given configuration of ridges. The need thus has arisen for some experimentally-determined design data, and it is the purpose of this paper to discuss and summarize the results of a systematic series of measurements of the magnetic field produced by ridged pole-pieces which cover most of the configurations likely to be used, with mean fields up to 20 kG.

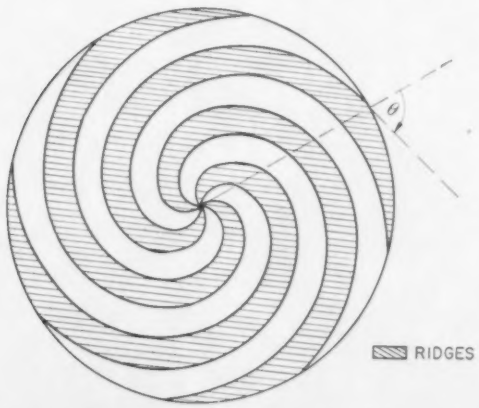

FIG. 1.—Example of a spiral ridge pole-piece (plan view).

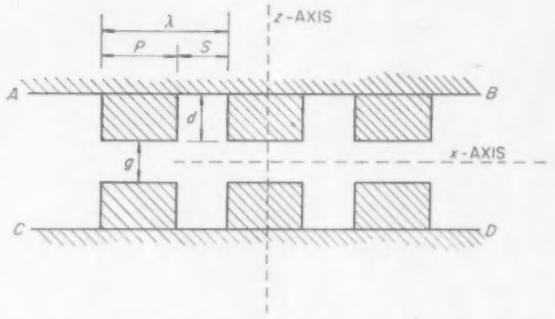

FIG. 2.—Cross-section of a system of rectangular ridges.

Although most of these measurements have been made on straight, parallel ridges of rectangular cross-section, as shown in Fig. 2, some measurements on spiral ridge pole-pieces shown that there is a simple correspondence with the parallel ridge results and that the latter are applicable directly to the design of spiral ridge pole-pieces.

The experimental measurements have been supplemented by a theoretical programme in which the method of conformal transformation has been used to calculate the field distribution to be expected from certain ridge configurations at low fields (when the steel surface is approximately a magnetic equipotential). The results of these computations provided

a check on the experimental values, and were used to assist in extrapolating the results outside the range covered by the measurements.

The parameters used are defined in Section 2. Section 3 describes briefly the experimental technique and treatment of observations and Section 4 describes the theoretical programme. The results are presented and compared with theory in Section 5, Section 6 describes methods of extrapolating the results and in Section 7 their applicability to the spiral ridge pole-piece is discussed. A brief account of the harmonic content of the field waveforms is given in Section 8 and finally, in Section 9, some remarks are included on the use of ridge cross-sections other than rectangular.

The use of a combination of ridges and pole-face windings to produce flutter fields is being investigated separately and is not discussed in this paper.

2. DEFINITION OF PARAMETERS

The minimum gap g, slot depth d, slot width s, ridge width p, and wavelength λ are defined in Fig. 2. A given two-dimensional ridge system is defined completely by the three independent parameters g/λ, d/λ, and p/λ.

An example of a spiral ridge system is shown in plan view of Fig. 1. Drawing a circle of radius R we define three additional parameters at this radius:

N = number of ridges crossing the circumference of this circle ($N = 4$ in Fig. 1)

θ = angle between the ridge and the radial direction

$\lambda_0 = 2\pi R/N$.

Using the co-ordinate system of Fig 2, the periodically varying field B on the median plane may be expressed as a Fourier series:

$$B = B_0[1 + \delta_1 \cos(2\pi x/\lambda) + \delta_2 \cos(4\pi x/\lambda) + \ldots] \quad (1)$$

where B_0 is the mean field.

It has been shown that the focussing properties of such a field are to a first approximation governed by the 'flutter' δ defined by

$$\delta^2 = \sum_{r=1}^{r=\infty} \delta_r^2 \quad (2)$$

When designing a complete pole-piece for a spiral ridge cyclotron the mean field must vary with radius in a specified manner. Therefore, we need to know how the effective mean gap G depends on g/λ, d/λ, p/λ, and B_0. For a given ridge system G would ideally be defined as the gap between two flat, parallel pole-pieces which would give the same mean field as the ridge configuration for the same energizing current in the magnet. Defined in this way, however, it is difficult to determine G experimentally and, for practical purposes, we use the following alternative definition:

$$G = (\int B\, dz \text{ across maximum gap})/B_0 \quad (3)$$

In other words G is the M.M.F. between lines AB and CD of Fig. 2 divided by B_0. This will always give a value close to the 'ideal' definition mentioned above, because lines AB and CD are approximately magnetic equi-potentials both at high fields (when the ridges are very saturated) and low fields (when all the steel is roughly at the same magnetic potential).

3. EXPERIMENTAL PROCEDURE AND TREATMENT OF OBSERVATIONS

For the two-dimensional measurements, the required arrangement of rectangular bars of mild steel was clamped to the pole-pieces of a large stabilized electromagnet of variable gap and mean field. The magnetic field distribution was determined with the aid of a search coil and an integrator designed on the principle of mutual inductance feedback (EDGAR, 1937). The instrument was calibrated in a field measured by nuclear resonance (GRAY, 1956) and compensated for thermo-electric drift. The accuracy in the 25 kG range was 0·05 kilogauss, being proportionately greater on lower ranges.

The measurement technique will not be described in detail here; adequate precautions were taken to eliminate errors due to fringe fields, misalignment of the search coil, and integrator drift.

To determine the mean gap G from equation (3), the line integral of the field across the gap was measured directly be replacing the search coil with a uniformly wound cylindrical coil, of the appropriate length ($g + 2d$), and calibrated in a known uniform field.

The field distribution produced by a given ridge system was measured, together with the mean gap G, for mean fields up to 20 kG. This procedure was repeated for a total of 72 different ridge systems covering the approximate range:—

$$0{\cdot}35 < p/\lambda < 0{\cdot}8$$
$$0{\cdot}1 < g/\lambda < 1{\cdot}0$$
$$0{\cdot}1 < d/\lambda < 0{\cdot}8$$

A mechanical Fourier analyser was used to find the harmonic coefficients of each field waveform, and δ was calculated using equation 1. Only the first three of four harmonics contributed significantly to δ. For each ridge geometry values of δ and G were determined by graphical interpolation for mean fields up to 20 kG at intervals of 2 kG.

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4. THEORETICAL PROGRAMME

At low fields, the surface of the steel approximates to a magnetic equi-potential and the method of conformal transformation (GIBBS, 1958) can be used to calculate the field distribution produced by a two-dimensional ridge system. In principle the low field values of δ and G can be found in this way for any two-dimensional array of rectangular cross-section, though in general this would involve a great deal of computation.

In the present investigation, however, the theory was only required to provide a check on some of the experimental values and to assist in extrapolating the results outside the range of geometric parameters covered by the measurements. Sufficient information for these purposes was obtained by solving two limiting cases:

(a) The case in which $p/\lambda = 0{\cdot}5$, $d/\lambda \to \infty$, and g/λ is variable. (Fig. 3a).

(b) The case of a single finite slot in each pole, with g, s, and d variable (Fig. 3b).

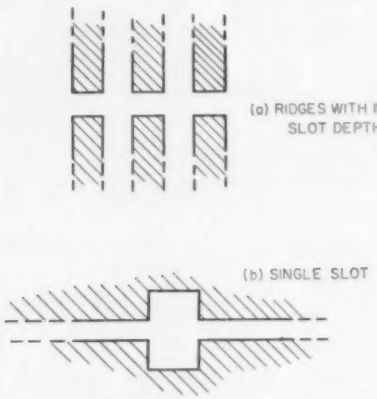

FIG. 3 (a, b).—Configurations analysed theoretically.

By analysing the resulting waveforms it was possible to obtain plots of δ and G/λ against g/λ for $p/\lambda = 0{\cdot}5$ and $d/\lambda = 0{\cdot}25$, $0{\cdot}5$ and infinity, the results being given in Section 5.

The solutions of cases (a) and (b) are given in the above reference and further details of this analysis will not be given here.

5. RESULTS

Some typical examples of experimental magnetic field waveforms are shown qualitatively in Fig. 4, and the variation of their harmonic content with mean field and geometry will be briefly discussed in Section 7. We shall proceed now to consider the two primary quantities of interest, δ and G. All the experimental values of these quantities have been tabulated elsewhere (SMITH, 1958) as a function of mean field and

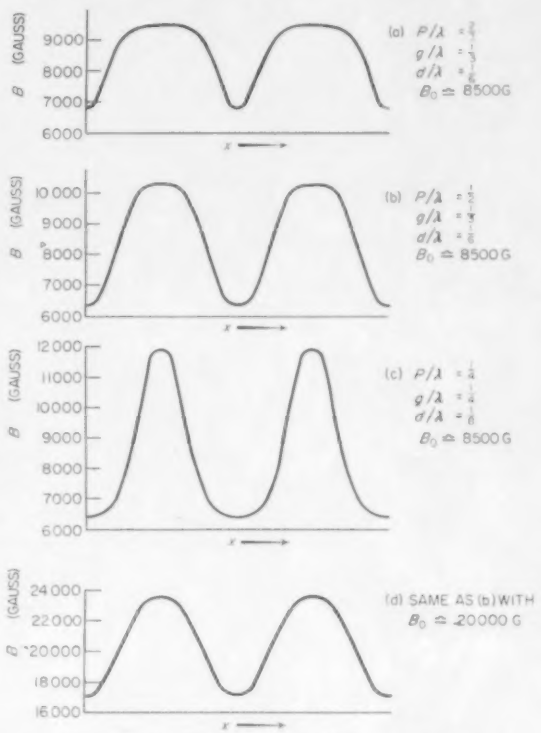

FIG. 4 (a–d).—Some examples of flutter waveforms.

ridge geometry, and reference should be made to these tables if maximum accuracy is required. For most purposes, however, the graphical summary of the results, given in this section, is sufficiently accurate.

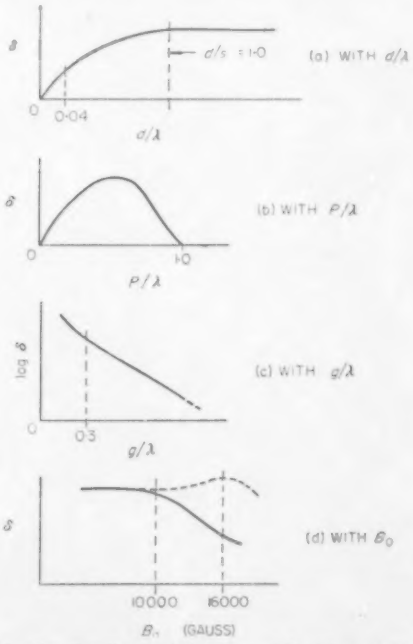

FIG. 5 (a–d).—Qualitative behaviour of δ with the individual parameters.

5.1 *Flutter values*

(a) *Qualitative behaviour.* The general way in which δ varies separately with d/λ, p/λ, g/λ and B_0 is shown qualitatively in Fig. 5. For small slot depths δ increases linearly with d/λ; when d/λ reaches about 0·04 the curve begins to flatten off until eventually a point is reached at which further increase in d/λ has practically no effect on δ, the curve becoming horizontal. It will be seen later that it is sometimes more useful to consider the variation of δ with d/s rather than d/λ. This curve is similar to Fig. 5(a), but shows that the horizontal portion commences approximately when $d/s = 1$.

The illustrated variation of δ with p/λ is to be expected from the fact that δ must obviously be zero, both when $p/\lambda = 0$ and when $p/\lambda = 1{\cdot}0$.

For mean fields up to about 6 kG δ remains constant, the surface of the steel closely approximating to a magnetic equi-potential. Saturation effects usually begin to be evident in the region $B_0 = 6$ to 10 kG and at these and higher fields δ normally decreases as B_0 increases. However, for values of p/λ greater than about 0·65, δ behaves as shown by the dotted curve in Fig. 5(d). This rise and fall is explained satisfactorily by the fact that the corners of the ridge saturate before the ridge as a whole.

The variation of δ with g/λ exhibits a remarkable consistency not shown by the other parameters, for it is found that experimental graphs of log δ against g/λ are approximately the same shape for all values of d/λ, p/λ, and B_0 covered by the measurements. For $g/\lambda > 0{\cdot}3$, all these graphs follow closely the relationship:

$$\delta \propto \exp(-\pi g/\lambda) \qquad (4)$$

For $g/\lambda < 0{\cdot}3$ the curve becomes somewhat steeper and the shape less constant. Equation (4) has a theoretical basis for certain limiting values of the other parameters and for large g/λ (see Section 6), but its validity for values of g/λ down to 0·3 would appear, to some extent, to be fortuitous.

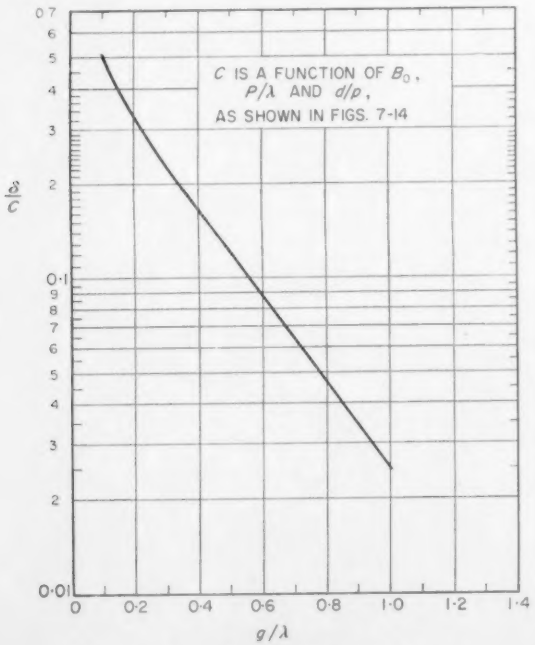

FIG. 6.—Variation of flutter with λ/g.

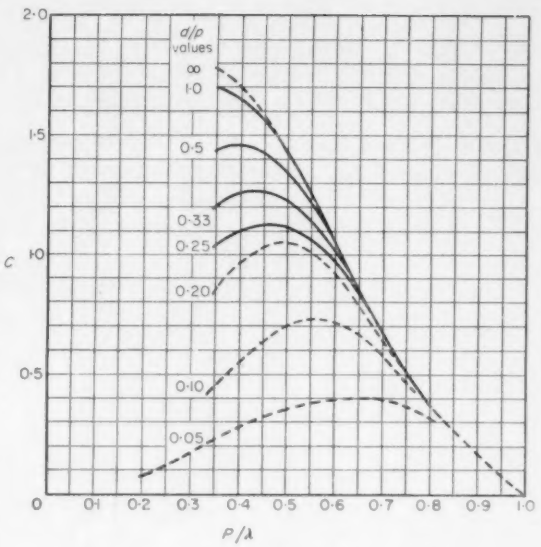

FIG. 7.—Values of C, for $B_0 = 6$ kG.

(b) *Graphical presentation.* Introducing a new parameter C, independent of g/λ the δ values may be conveniently presented by means of a single graph of δ/C against g/λ (see Fig. 6), together with a series of graphs giving the values of C as a function of p/λ, d/p and B_0 (see full line curves in Figs 7 to 14). The parameter d/p occurs here instead of d/λ for reasons of experimental convenience and can, of course, easily be converted to the latter by multiplying by p/λ. The dotted curves are those obtained by extrapolation (see Section 6).

The values of C have been adjusted to give the best fit for values of g/λ near 0·5. For $g/\lambda < 0{\cdot}3$ the shape of the curve varies somewhat and this simplified presentation may result in a decrease in accuracy in this region.

(c) *Comparison with theory.* A direct comparison between the calculated and experimental flutter values is given in Table 1. There was no significant difference between the computed wave-forms for $d/\lambda = 0{\cdot}5$ and $d/\lambda = \infty$, and hence the flutter values for these two cases are the same. The experimental values of δ are

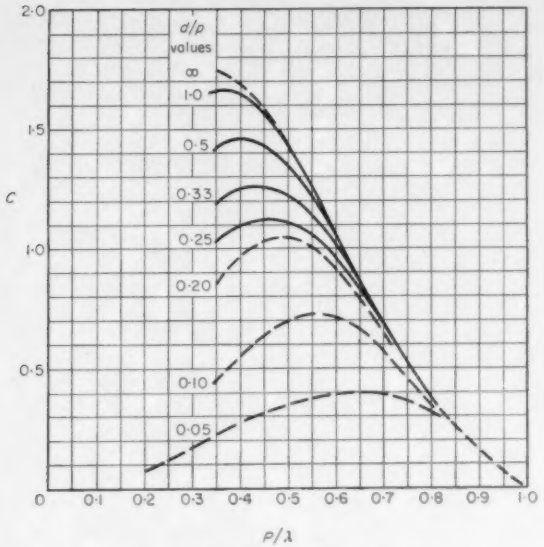

FIG. 8.—Values of C, for $B_0 = 8$ kG.

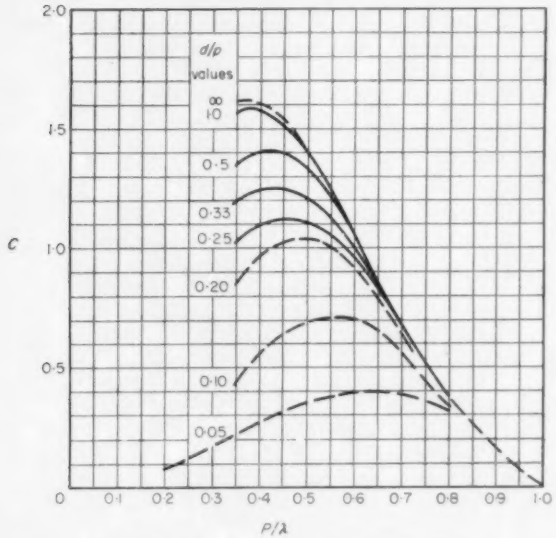

FIG. 9.—Values of C, for $B_0 = 10$ kG.

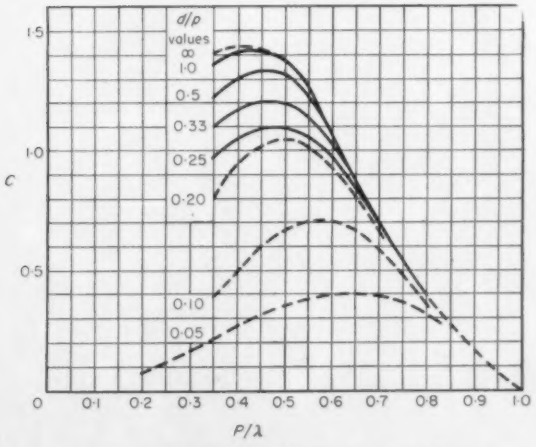

FIG. 10.—Values of C, for $B_0 = 12$ kG.

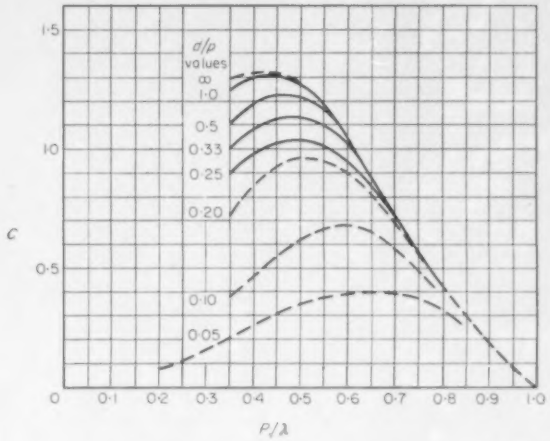

FIG. 11.—Values of C, for $B_0 = 14$ kG.

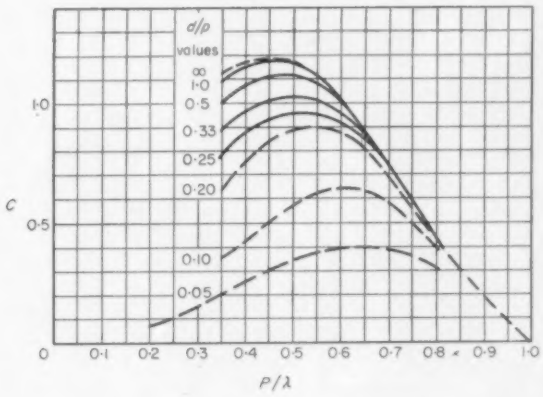

FIG. 12.—Values of C, for $B_0 = 16$ kG.

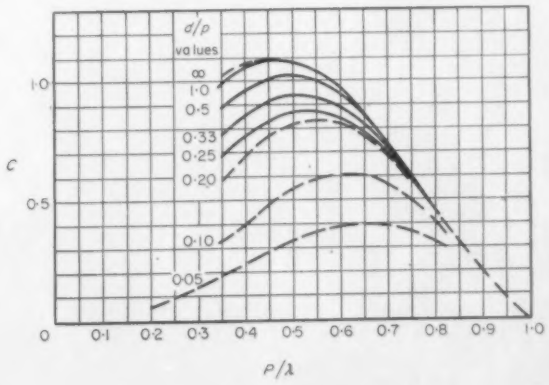

FIG. 13.—Values of C, for $B_0 = 18$ kG.

TABLE 1.—COMPARISON BETWEEN THEORETICAL (INFINITE μ) AND EXPERIMENTAL ($B_0 = 6000$ GAUSS) VALUES OF δ, FOR $p/\lambda = 0{\cdot}5$

Value of g/λ		0·05	0·1	0·167	0·25	0·333	0·507	0·704
$d/\lambda = \infty$ and $d/\lambda = 0{\cdot}5$	δ (Theory)	0·94	0·724	0·540	0·386	0·289	0·159	0·084
	δ (Experiment)	—	—	0·533	0·390	0·295	0·163	0·083
$d/\delta = 0{\cdot}25$	δ (Theory)	0·92	0·707	0·521	0·365	0·271	0·152	—
	δ (Experiment)	—	—	0·515	0·368	0·278	0·156	—

taken from the original tabulated values and not from the curves given in this section which, as indicated above, are slightly less accurate.

The agreement between experiment and theory is well within the limits of experimental error.

5.2 *G values*

(a) *Qualitative discussion.* The effective mean gap must always have a value between the minimum gap g and the maximum gap $(g + 2d)$. The amount by which the mean gap exceeds the minimum gap is the sum of two distinct contributions. The first of these, which we shall refer to as the 'geometric effect', is dependent on the geometric parameters of the ridge system. The second contribution, which we refer to as the 'saturation effect' results from the additional reluctance of the ridges due to the finite permeability of the steel. The latter contribution has the approximate value $2d/\mu$ where μ is an effective permeability which depends on the average flux density in the ridge itself.

Thus, we may write:

$$G = g + F_1(d/\lambda, p/\lambda, g/\lambda) + 2d/\mu \qquad (5)$$

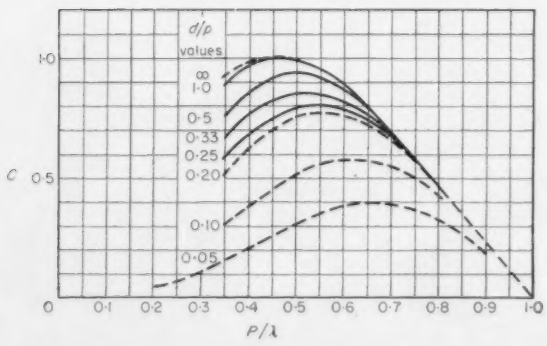

FIG. 14.—Values of C, for $B_0 = 20$ kG.

In practice F_1, the geometric effect, is found to depend somewhat on mean field, normally decreasing as B_0 increases. This is presumably due to the gradual straightening of the magnetic equi-potentials.

At low fields μ is very large and the geometric term predominates; from about 14 kG upwards the opposite is true and the geometric term is usually negligible compared with $2d/\mu$.

(b) *Graphical summary.* Between 14 and 20 kG it is found that the quantity $(G - g)/2d$ is to a good approximation dependent only on p/λ and B_0. The existence of this relationship, summarized in Fig. 15, shows that F_1 in (5) is usually negligible at these

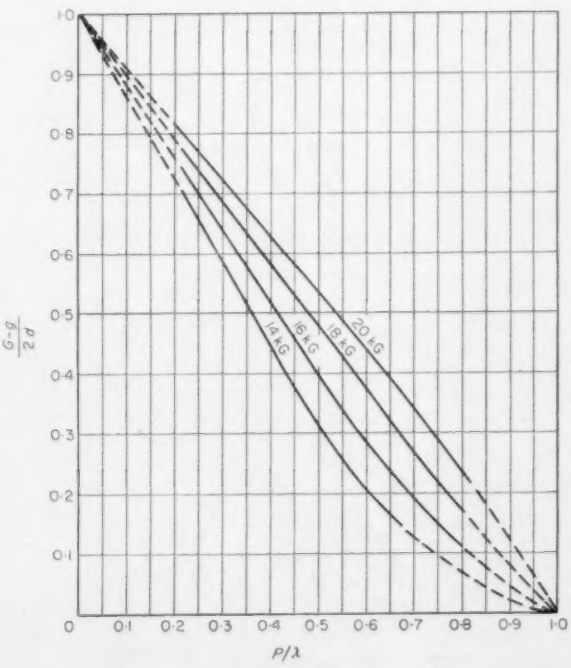

FIG. 15.—Summary of mean gap values at high fields.

TABLE 2.—COMPARISON BETWEEN THEORETICAL (INFINITE μ) AND EXPERIMENTAL ($B_0 = 6000$ GAUSS) VALUES OF G/λ, FOR $p/\lambda = 0{\cdot}5$

Value of g/λ		0·05	0·1	0·167	0·25	0·333	0·507	0·704
$d/\lambda = \infty$ and $d/\lambda = 0{\cdot}5$	G/λ (Theory)	0·083	0·149	0·230	0·323	0·411	0·588	0·788
	G/λ (Experiment)	—	0·17	0·25	0·34	0·425	0·595	0·81
$d/\lambda = 0{\cdot}25$	G/λ (Theory)	0·082	0·148	0·227	0·319	0·409	0·585	—
	G/λ (Experiment)	—	0·17	0·25	0·34	0·425	0·595	—

mean fields. Values of G derived from Fig. 15 are in most cases within 4 per cent of the tabulated values (SMITH, 1958). For the portions of the curves shown dotted at the high p/λ end, however, the geometric term is still significant and the values of G calculated from the curves in this region consequently are rather less accurate. Although the measurements only covered p/λ values down to about 0·3, the curves have been extended back to the obvious point of convergence ($p/\lambda = 0$, $(G - g)/2d = 1$).

Below 14 kG the geometric and saturation effects become comparable in size and G is found to depend on the four parameters g/λ, d/λ, p/λ and B_0 in a very complex manner. This dependence exhibits no apparent simplifying features and a large number of graphs are required to present the results in this region. These will not be given here and reference should be made to the tabulated values (SMITH, 1958) for G values between 6 and 14 kG. At 6 kG and below, however, when the geometric effect predominates, an estimate of G can often be obtained from the conformal transformation computations and these are discussed in the next paragraph.

(c) *Comparison with theory*. A direct comparison between the 6 kG experimental values of G/λ and the computed values is shown in Table 2 for $p/\lambda = 0{\cdot}5$ and $d/\lambda = 0{\cdot}5$ and 0·25. In all cases, the experimental values were slightly higher than the theoretical values. This indicates that the effects of finite permeability are present even at low fields, though they are not sufficient to affect δ.

The conformal transformation analysis also shows that as the slot depth increases, the quantity $(G/\lambda - g/\lambda)$ tends to a limiting value which is a function of p/λ and, to a much smaller extent, of g/λ. These limiting values, calculated assuming infinite permeability, are summarized graphically in Fig. 16 and hold

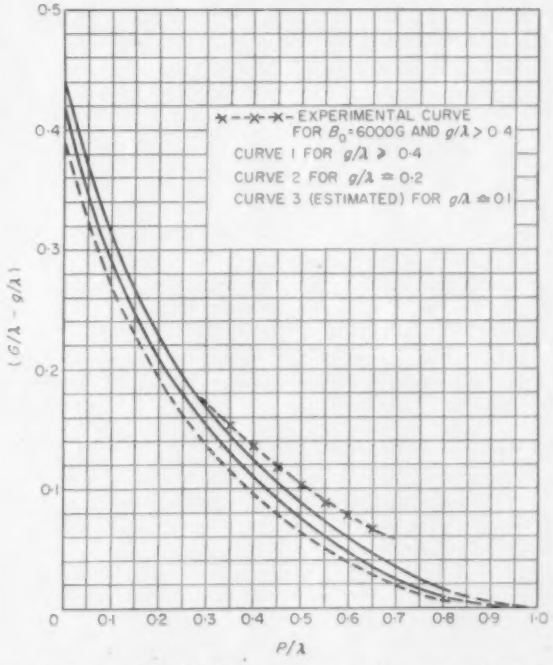

FIG. 16.—Theoretical limiting values of $(G/\lambda - g/\lambda)$, for large slot depth and infinite permeability.

with sufficient accuracy for values of $d/s \geqslant 0{\cdot}8$. A comparison between the upper curve of Fig. 16 and the corresponding 6 kG experimental values of $(G/\lambda - g/\lambda)$ shows that the effect of finite μ is very evident at high p/λ, presumably due to saturation effects at the corners of the ridges. In practice the curves would also be unreliable for very small values of p/λ, since very thin ridges would begin to saturate at relatively low fields.

5.3 *Errors*

The sources of error, which will not be discussed in detail, may be broadly divided into errors of measurement, errors of harmonic analysis and errors of

interpolation. The treatment of observations was designed to minimize the accumulation of these errors. The accuracy of the δ values obtained from the graphs in this report varies with mean field and geometry, but in most cases the error will not exceed 2 per cent or 0·005, whichever is greater. The error in values of G obtained from Fig. 15 will in most cases not exceed 4 per cent.

6. EXTRAPOLATION OF RESULTS

Preliminary design calculations indicate that most ridge geometrics likely to be used lie within the range covered by the measurements. However, since it may occasionally be necessary to estimate δ or G for ridge systems outside this range, the extrapolations of the results will be discussed in some detail.

6.1 *Flutter values*

(a) *High* g/λ. From the generality of equation (4) and its theoretical basis, it may safely be assumed that the graph of Fig. 6 may be extended linearly to higher values of g/λ.

(b) *Low* g/λ. For $g/\lambda < 0{\cdot}3$ the shape of the curve of Fig. 6 is no longer constant, but varies in a complicated way with the other parameters; a reasonable extrapolation in an individual case may be obtained by calculating the limiting δ as $g/\lambda \to 0$. If saturation effects are small then as $g/\lambda \to 0$ the waveform becomes rectangular with zero minimum field, the harmonics for which may be readily calculated. (e.g. When $p/\lambda = 0{\cdot}5$, δ approaches the values 1·41 as $g/\lambda \to 0$.)

(c) *High* p/λ. When $p/\lambda = 1{\cdot}6$, 0 must be zero. The completion of the curves in this direction thus presents no difficulty, and are shown by dotted lines in Figs. 7 to 14.

(d) *Low* P/λ. As the ridges become thinner, the increasing saturation of the steel results in the flutter tending to zero as $p/\lambda \to 0$, for all values of the other parameters. Since most of the curves of Figs. 7 to 14 have already passed their maxima, their extension down to the point (0, 0) may be estimated with reasonable accuracy, though this has not been done here. Theoretical values of δ are of little use in this region, as the saturation effects at low p/λ decrease δ considerably, even at low fields.

(e) *High* d/λ. As stated earlier further increase of slot depth beyond the value $d/s = 1$ has practically no effect on δ. For values of $p/\lambda < 0{\cdot}5$, the measured values did not quite extend to this point, but it was estimated with the aid of the fact that a graph of log C against log (d/s) has nearly the same shape for all p/λ and B_0. The limiting values of C for effectively infinite slot depth obtained in this way are shown as dotted curves in Figs. 7 to 14.

(f) *Small* d/λ. For values of d/λ below 0·1, additional experimental and theoretical data is available. Measurements have been made by BLUEMEL *et al.*, (1955) on spiral pole pieces with $p/\lambda = 0{\cdot}5$, g/λ between about 0·2 and 1·0 and with most of the values of d/λ less than 0·08. For these shallow ridges the effects of saturation were very small, and δ was found to be independent of mean field up to about 17 kG. All their results were found to be in good agreement with their theoretical formula which, in our notation, is:

$$\delta^2 = 16(d/\lambda)^2\left[\left(\sinh\frac{\pi(g+d)}{\lambda}\right)^{-2} + \left(\sinh\frac{3\pi(g+d)}{\lambda}\right)^{-2} + \ldots\right] \quad (6)$$

as $g/\lambda \to 0$ this correctly gives $\delta = \sqrt{2}$, the flutter for a rectangular wave of $p/\lambda = 0{\cdot}5$; while for $g/\lambda > 0{\cdot}5$ it approximates to the following expression, which may also be obtained by conformal transformation, assuming $d/\lambda \ll 1$:

$$\delta \simeq 8(d/\lambda)\exp(-\pi g/\lambda) \quad (7)$$

The expression may be generalized to include the parameter p/λ. In general:

$$\delta^2 = 16(d/\lambda)^2 \sum_{n=1}^{n=\infty}\left(\frac{\sin n\pi p/\lambda}{\sinh n\pi(g+d)/\lambda}\right)^2 \quad (8)$$

Usually only the first two or three terms of this summation will be significant and when $g/\lambda > 0{\cdot}5$, equation (8) approximates to:

$$\delta \simeq 8d/\lambda \exp(-\pi g/\lambda)\sin(\pi p/\lambda) \quad (9)$$

These formulae should be reliable for values of d/λ up to 0·04 and for mean fields up to, say, 16 kG. With their aid δ values for small slot depths have been estimated, and are shown as dotted curves in Figs. 7 to 14. For mean fields above 16 kG and for values of d/λ between 0·04 and the experimental region, the above mentioned relationship between log c and log d/s was used.

(g) *Low* B_0. All δ values below 6 kG are essentially the same as those quoted for $B_0 = 6$ kG. This is confirmed by a few measurements made on ridge systems at low fields, and by the agreement between the 6 kG values and the computed values of δ.

(h) *High* B_0. In the region 16 to 20 kG, the relationship

$$\delta \propto 1/B_0 \quad (10)$$

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is found to hold approximately. Equation (10) may thus be used to estimate values of δ somewhat higher than 20 kG. This extrapolation process should not proceed beyond, say, 24 kG, as δ soon begins to fall off more rapidly than given by (10).

6.2 *Mean gap values*

We saw in Section 5 that the geometric effect is usually negligible above 14 kG and equation (5) becomes:

$$G \simeq g + 2d/\mu \tag{11}$$

For mean fields up to 20 kG, therefore, values of $(G - g)/2d$ obtained from Fig. 15 may be used to calculate G for values of d/λ, p/λ and g/λ outside the range covered by the measurements. This method should not be used for small values of d/λ ($<0{\cdot}1$, say), which have been shown (BLUEMEL *et al.*, 1955) to be free of saturation effects up to about 17 kG.

For mean fields above 20 kG, G can be estimated from (11) by calculating from the normal induction curve of the material on the assumption that all the flux passes through the ridge. μ is then the permeability corresponding to a flux density of $\lambda B_0/p$ gauss.

Below 14 kG no simple extrapolation technique is available, though at low fields the curves of Fig. 16 may be used, with due regard to their limitations which are discussed in Section 5.2(c).

6.3 *Effect of composition of the steel*

The normal induction curve of the steel used to make the ridges is shown in Fig. 17. The results of the measurements apply accurately only to ridges whose material has approximately the same characteristic. Differences in permeability will affect G to an extent which may be estimated from the discussions in Sections 5.2 and 6.2. The effect on δ is not easy to estimate but a tentative calculation suggests that a decrease of 1 kG in the 'saturation flux density' might decrease δ from, e.g., 0·25 to 0·24 at 16 kG mean field.

7. SPIRAL RIDGES

We shall now show how the results for straight parallel ridges may be used to predict the 'flutter' and mean gap values for a three dimensional spiral ridge system.

First we define the wavelength λ at any radius R by:

$$\lambda = \lambda_0 \cos\theta \tag{12}$$

where $\lambda_0 = 2\pi R/N$ and θ is the spiral angle at radius R, as defined in Section 2.

Secondly we denote the width of a ridge, measured around the circumference of a circle of radius R, by p_0. The definitions of d and g remain unchanged.

At any radius of the spiral ridge system, therefore, we may regard the quantities $(g/\lambda_0 \cos\theta)$, $(d/\lambda_0 \cos\theta)$ and p_0/λ_0 as equivalent to g/λ, d/λ and p/λ respectively, and using these values we may determine δ and G from the parallel ridge results. The justification for this procedure will now be discussed:

By examining Fig. 1 it may be seen that when θ is large (at the larger radii) the curvature of the ridges is small, and in a given region the ridges are nearly straight and parallel. The two dimensional results would therefore be expected to hold in this region, and this has been confirmed by some unpublished measurements by T. C. Randle using the pole-piece of Fig. 1.

Towards the centre of the pole-piece, however, θ must decrease, until the ridges converge radially into the centre. To investigate the behaviour of δ and G in this region, measurements using the pole-pieces shown in Fig. 18 and 19 were included in our experimental programme. The variation of δ with radius for the pole-piece of Fig. 19 was predicted from the parallel ridge data using the procedure outlined above,

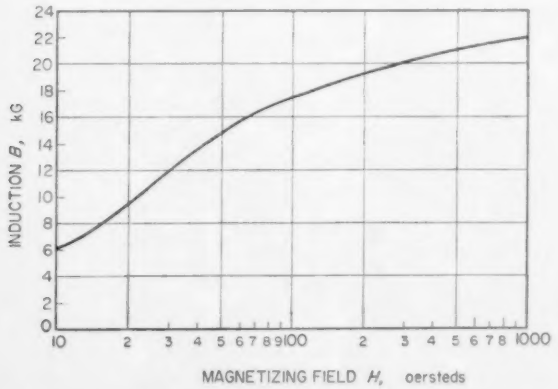

FIG. 17.—Normal induction curve of the mild steel used in the measurements.

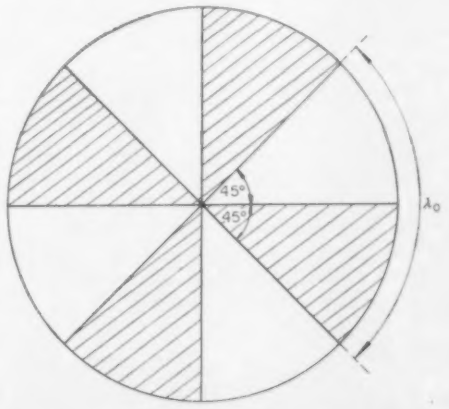

FIG. 18.—Experimental radial ridge system.

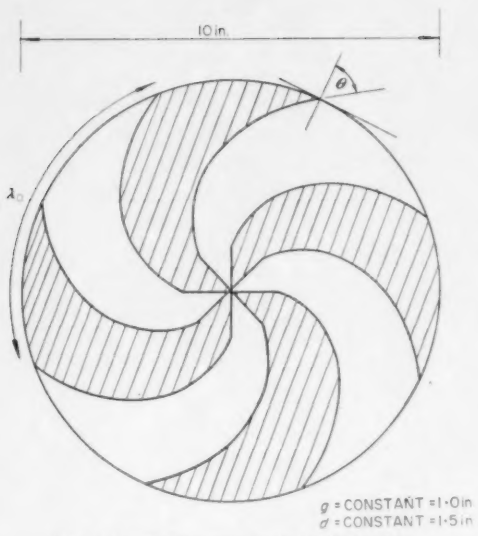

FIG. 19.—Experimental spiral ridge system.

and the agreement between measured and predicted values is shown in Fig. 20.

Similar agreement was found for the pole-piece of Fig. 18, and in both cases the measured values of G were within a few per cent of the predicted values.

Towards the outside edge of the pole-piece, however, fringing effects will cause δ to depart from its predicted value. In the absence of correcting shims this effect should become noticeable at a distance of about $(g/2 + d)$ from the edge and since the field minimum will be affected more than the field maximum, the initial effect will be an increase in δ. This, also, has been confirmed by T. C. Randle.

8. HARMONIC CONTENT OF THE WAVEFORMS

When analysing the particle motion in spiral ridge magnetic fields it may occasionally be of value to know the individual values of the harmonics which contribute to the overall flutter value. Apart from the first harmonic B_1, which is obviously the largest, only the 2nd and 3rd and 4th harmonics usually are significant. In general, the dependence of these higher harmonics on mean field and geometry is so irregular that an adequate discussion of their behaviour is impossible. For a given ridge system, a harmonic may even change sign as the mean field is increased.

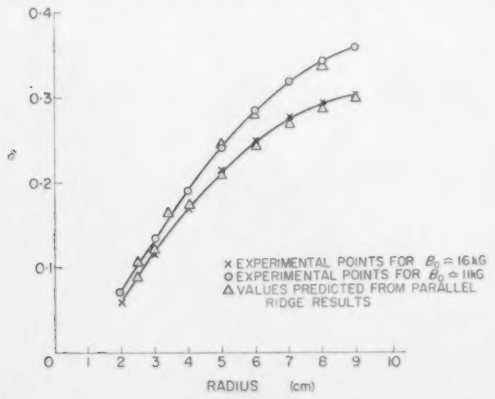

FIG. 20. Comparison between predicted and experimental δ values for the spiral ridges of Fig. 19.

If we confine our attention to the results for $p/\lambda = 0{\cdot}5$, however, we find that only the 2nd and 3rd higher harmonics, B_2 and B_3, are significant, and when expressed as a fraction of the 1st harmonic are, to a first approximation, independent of d/λ. Accordingly the ratios B_3/B_1 and B_2/B_1 have been plotted against g/λ for different mean fields in Figs. 21(a) and 21(b). The values obtained from the theoretical (infinite permeability) curves are also plotted for comparison purposes.

The ratio B_3/B_1 is found to be independent of mean field, and to agree well with the theoretical points. The values of B_2/B_1, however, are always lower than the theoretical values, even at low fields. This provides a very elegant confirmation of the fact that saturation effects occur at the corners of the ridges at low fields, which would reduce the even harmonics in the curve, but not the odd harmonics.

9. ALTERNATIVE RIDGE CROSS-SECTIONS

Although this account has been devoted exclusively to ridges of rectangular cross-section, which in practice would be the simplest to manufacture, it is of interest to enquire whether there would be any advantage in using a different cross-section (e.g. trapezoidal).

This question has not been examined in any detail, but a few approximate measurements have been made which indicate that for given values of g and d the flutter values are usually somewhat smaller for cross-sections other than rectangular. Thus they are of little interest except in the particular case of ridges shaped to produce a pure sinusoidal field on the median plane. Since in some applications it may be necessary to produce a flutter field free of higher harmonics, this case has been studied theoretically by WILKINS (1956) from whose equations one can calculate the upper limit to δ for a given g/λ (assuming no saturation effects in the steel). The resulting limiting curve of δ against g/λ approximately coincides with the curve of Fig. 6 when C has the value 0·98. Examination of Figs. 7 to 14 shows that values of C up to at least 1·5 are obtainable with rectangular ridges, showing a gain of about 50 per cent over the flutter obtainable with 'sinusoidal' ridges.

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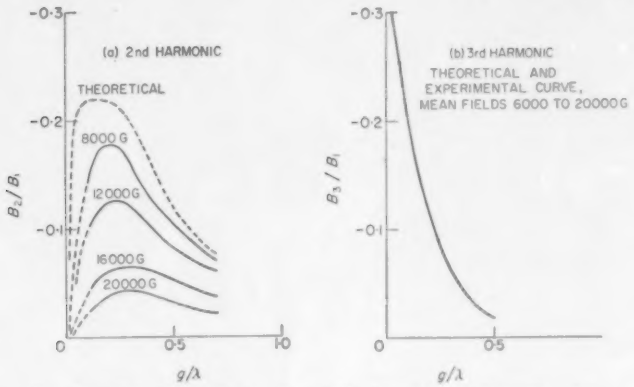

FIG. 21.—Behaviour of second and third harmonics, for $p/\lambda = 0{\cdot}5$.

An example of the cross-section required to produce a pure sinusoidal field is shown approximately in Fig. 22. Measurements made using a set of ridges with this cross-section show that the smooth contour of the cross-section results in saturation characteristics different from those observed with rectangular ridges. Fig. 23 shows plots of δ against B_0 for

(a) sinusoidal ridges,
(b) rectangular ridges with the same value of δ at low fields, and
(c) rectangular ridges with same values of g/λ and d/λ as the sinusoidal ridges.

It may be seen that sinusoidal ridges initially saturate more slowly than rectangular ridges, but for a given g/λ and d/λ the latter yield a larger flutter at all field levels.

10. CONCLUSIONS

The field distribution produced by parallel ridges of rectangular cross-section has been measured for a wide range of geometric configurations and mean fields. The accuracy of the results has been confirmed, and their range extended with the aid of computed field distributions.

Measurements on spiral ridges, confirmed by other workers, show that the parallel ridge results may be used with confidence to design spiral ridge magnets even when the spiral angle is large.

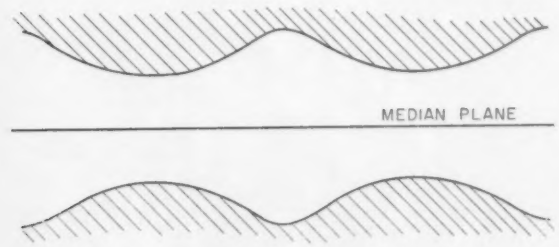

FIG. 22.—Example of a ridge cross-section giving a pure sinusoidal field distribution.

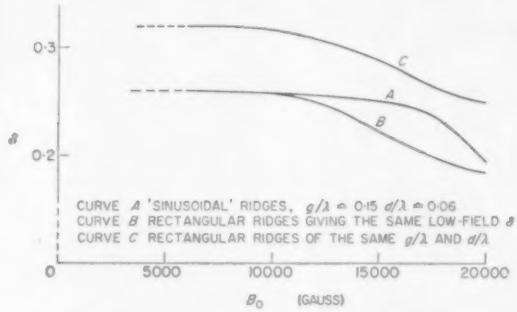

FIG. 23.—Comparison of saturation effects for 'sinusoidal' and rectangular ridges.

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Mr. E. B. FOSSEY of the Theoretical Physics division was responsible for most of the computational work.

Mr. D. A. GRAY designed and constructed the magnet stabilizer and the nuclear resonance equipment.

Technical assistance was provided by B. R. ADAMS, W. BROWN, V. EDWARDS and Miss S. ALLUM.

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WILKINS J. J. (1956) *Proc. C.E.R.N. Symposium on High Energy Accelerators*, p. 20.

J. Nucl. Energy, Part C: Plasma Physics, 1959, Vol. 1, p. 66. Pergamon Press Ltd. Printed in Northern Ireland

BOOK REVIEW

Accelerators of Ions and Electrons, Č. Šimoně. Artia (Prague) and Constable (Gt. Britain) 191 pp. 16s.

THERE is no date on the title page of this book, and though it has presumably only recently been translated, it appears to have been written about seven years ago; thus the start-up of the Cosmotron at Brookhaven in May 1952 is reported, but the 'strong-focusing' principle discovered there later in the same year is not. Nevertheless, the book forms a very readable introduction both to accelerator principles and to practical techniques, especially as applied to small machines.

Electrostatic accelerators, betatrons, synchrotrons and cyclotrons are well covered; the section on high-frequency linear accelerators however is out of date even for 1952, and the description of the Alvarez proton accelerator is incorrect.

There are over a hundred diagrams, including a number of photographs. The diagrams are clear and adequate, but the photographs are of poor quality. Several particular machines are illustrated and described without reference to their location; thus no indication is given that the "Proton Synchrotron for $1{\cdot}3 \cdot 10^9$ eV" shown on p. 120 is in Birmingham, or even that it is in England.

The translation is by no means perfect, and a number of technical terms are wrongly used, for example the term 'electron jet' is used instead of 'electron gun'; serious confusion is unlikely, however.

The preface makes strange reading, especially to Western readers. Here we find for example that "Engels says 'The electron is as inexhaustible as the atom', and science today fully confirms his words and will confirm them constantly".

Despite its faults, however, this book will serve as a useful introduction to accelerators for those interested in practical details and techniques as well as the theory.

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J. Nucl. Energy, Part C: Plasma Physics, 1959, Vol 1, pp. 67 to 85. Pergamon Press Ltd. Printed in Northern Ireland

ABSTRACTS FROM THE PHYSICS OF PLASMAS*

PREFACE

THE present collection consists of articles in which are presented the theoretical and experimental investigations carried out in 1951–1958 in the Institute of Atomic Energy of the USSR Academy of Science on the problem of the controlled thermonuclear reaction and on questions of the physics of plasma associated with it. Only those articles which have not been published previously in the press are included in this collection. The collection is divided into four volumes and the articles are presented generally in chronological order.

The editorial work in this collection was carried out by the candidate of physical mathematical sciences, V. I. KOGAN. The responsible editor was Academician M. A. LEONTOVICH.

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VOLUME 1

TAMM I. E., *Theory of the magnetic thermonuclear reactor, Part* **1**, pp. 3–19 (completed 1951)†.

This basic article can be best characterized by giving the headings of the various sections:

(1) The motion of a charged particle in crossed fields.
(2) Kinetic equations. First approximation theory. (Neglect of collisions).
(3) Second approximation theory, the role of collisions. Heat conductivity and viscosity of the plasma.
(4) Preliminary calculation of a small model of a controlled thermonuclear reactor with heating at a temperature discontinuity.

SAKHAROV A. D., *Theory of the magnetic thermonuclear reactor Part* **2**, pp. 20–30 (completed 1951).

The previous article by I. E. TAMM presented the properties of the high temperature plasma in a magnetic field and gave hope for realizing the controlled thermonuclear reactor. In this article other problems in the theory of the controlled thermonuclear reactor are presented and in particular:

(1) Thermonuclear reactions, bremsstrahlung.
(2) Calculation of a large model. Critical radius. Boundary phenomena.
(3) The intensity of magnetization. Optimal construction. Productivity with respect to active species.
(4) Drift in a non-homogeneous magnetic field, and increased current. Induction stabilization.
(5) The problem of the instability of the plasma.

The article contains some practical details, such as what temperatures and alloys are to be used, and the proposed dimensions of the apparatus.

TAMM I. E., *Theory of the magnetic thermonuclear reactor Part* **3**, pp. 31–41.

The sub-heading of this article is: The Drift and Heat Conductivity of the Plasma in a Toroid in the Presence of a Stabilized Current. This work was carried out with the help of N. N. BOGOLYUBOV.

Formulae are given which permit the heat conductivity to be calculated as a function of the field of the stabilized current flowing in the plasma, and of the field of the windings on the toroid.

ANDRIANOV A. M., BASELEVSKAYA O. A., LUK'YANOV S. YU., OSOVETS S. M., PETROV YU. F., PODGORNI I. M. and YAVLINSKI N. A., *Investigation of the heating of hydrogen plasma in small toroidal systems*, pp. 42–65 (completed 1951).

The physical basis of the optical and electrical method of measuring some of the parameters of the plasma are described. The experimental set-up is given and also the technique of carrying out the measurements. The more important experimental data are given and the results obtained discussed.

BUDKER G. I., *Problems associated with the drift of particles in a toroidal magnetic thermonuclear reaction chamber*, pp. 66–76 (completed 1951).

An investigation is made of the drift of particles in a non-homogeneous magnetic and electric field. An evaluation is made of the heat conductivity in a torus, on stabilization of the drift by a magnetic field of a longitudinal current, and by some other methods. The considerations presented, lead to the conclusion that the compensation of the drift by weak longitudinal current is insufficient. An evaluation shows that it is necessary for the realization of a thermonuclear reaction that the field of the stabilizing current be of the same order of magnitude as that of the

* U.S.S.R. Academy of Sciences. Abstracts prepared for the U.S.A.E.C.

† Here and subsequently the date refers to the date of completion of the work.

basic field. In addition to this, it is necessary to create additional fields associated with the walls of the chamber which will hold the plasma as a whole within the torus.

DAVIDOV B. I., *The influence of plasma oscillations on the electrical and thermal conductivity of plasma*, pp. 77–88 (completed 1951).

The influence of plasma oscillations on the motion of electrons and ions and on plasma heat conductivity is evaluated assuming that the oscillation amplitude corresponds to heat equilibrium. It is well known that any ordered streams of currents in the plasma lead to a build-up of oscillations. In addition, their intensities may be considerably greater than that given by the Rayleigh-Jeans formula.

DAVIDOV B. I., *Breakdown mechanism for an electrodeless discharge*, pp. 89–94 (completed 1951).

After a general discussion of the difference between breakdown with and without electrodes, the author considers the cases of breakdown without, and then with a constant longitudinal magnetic field.

ANDRIANOV A. M. and OSOVETS S. M., *Theory and experiments on the starting of an electrode-less discharge in a magnetic field*, pp. 95–109 (completed 1951).

The topics in this article are:

(1) Theory of breakdown in the absence of the magnetic field.
(2) Theory of breakdown in a torus, or in the presence of a longitudinal magnetic field.
(3) Time dependence.
(4) Experiments for studying the breakdown in hydrogen. The work was done in a glass tube with a diameter of 4 cm bent in the form of a torus of a diameter of 50 cm. The electric field inside the torus is produced by the discharge of 2·5 μF through a copper coil placed in the middle of the torus. Inside the windings, for increasing the coupling between the copper windings and the gas, there is an iron insert with a 150 cm^2 cross-section. Tables are given of the results as a function of pressure and potential.
(5) Conclusions show that the experimental results agree within 30 per cent of the theory.

LEONTOVICH M. A., *Forces acting on a linear current which is within a conducting cylindrical tube*, pp. 110–114 (completed 1951).

In connexion with the problem of stabilizing the currents using a conducting tube surrounding it, several elementary formulae for forces acting on the current have been derived.

BRAGINSKI S. I., *Compression of a plasma under the action of its own magnetic field*, pp. 115–121 (completed 1951).

Preliminary results are given of the study of the stationary problem of the compression of a plasma in an infinitely long pipe. The plasma is considered completely ionized. The temperature of electrons and ions is assumed to be the same. The external magnetic field is absent. A more detailed investigation also for the non-stationary processes has been carried out by the author subsequently and published in *Journal of Experimental and Theoretical Physics* **33**, 645 (1957).

BUDKER G. I., *The betatron method of heating plasma to high temperatures*, pp. 122–129 (completed 1951).

A new method of heating plasma to high temperatures—the so called betatron method of heating—is based on the fact that on increasing the magnetic field the temperature of the plasma increases. The results are compared with the previously proposed method of heating using a longitudinal electric field. The problem is solved for the first model: an infinite straight tube with a magnetic field directed along the axis.

KOGAN V. I., *The rate of equilization of the temperatures of charged particles in a plasma*, pp. 130–137 (completed 1951).

Calculations are made in this work of:

(1) The rate of energy exchange between two degrees of freedom of a gas of charged particles present at one temperature, and a third degree of freedom of the same gas present at another temperature (applicable to the problem of the betatron heating of the plasma).
(2) The velocity of equalization of the temperature of charged particles of the two types for any ratio of their masses.

ZUBAREV D. N. and KLIMOV V. N., *The theory of the temperature discontinuity at the boundary of a plasma in a magnetic field*, pp. 138–161 (completed 1951).

The temperature discontinuity has been examined at the boundary of a magnetized plasma taking into account processes of ionization, excitation and charge transfer of atoms in the layer near the walls.

It is shown that the temperature discontinuity can take place under the conditions prevailing on the large model of a controlled thermonuclear reactor

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(the mean free path of neutral particles being small compared with the dimensions of the system) for values of heat flow to the walls, and for particle densities in the centre of the system of the order of magnitude of those in the model.

GALITSKI V. M. and MIGDAL A. B., *Dielectric constant of a high temperature magnetized plasma and the evaluation of the radiative heat conductivity*, pp. 161–171 (completed 1951).

An expression is found for the dielectric constant of a plasma situated in a magnetic field at high temperatures, when there are relativistic effects present. On the basis of the imaginary part of the dielectric constant, an evaluation is made of the radiative thermal conductivity. The radiative thermal conductivity is found to be smaller than the thermal conductivity produced by the particles.

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KOGAN V. I. and MIGDAL A. B., *The intensity distribution in the bremsstrahlung spectrum as a function of electron temperature*, pp. 172–177 (completed 1951 and revised later in preparation for publication).

Approximate expressions are derived for the dependence of the intensity distribution in the bremsstrahlung spectrum on the electron temperature in a hydrogen plasma (in the Born and quasi-classical regions of temperature). The possibility of using these expressions for the experimental determination of electron temperatures is discussed.

BRAGINSKI S. I., *The flow of particles and heat across a strong magnetic field in a completely ionized two-temperature plasma*, pp. 178–185 (completed 1952).

Electron conductivity of the plasma and the energy flow carried by electrons across a weak magnetic field has been calculated by LANDSHOFF (*Phys. Rev.* **76,** 904, 1949). Transverse flow of particles and heat in a strong magnetic field for the same temperatures of ions and electrons has been calculated by TAMM (page 3). The general transfer equations under these conditions has been obtained by FRADKIN (*Journal of Experimental and Theoretical Physics* **32,** 1176, 1957). The transfer equations for the case of different temperatures of ions and electrons can be obtained by a method BRAGINSKI (*J. exp. theor. Phys.* **33,** 459, 1957) analogous to the method of CHAPMAN and COWLING. The transverse currents in a strong magnetic field for different temperatures of electrons and ions has been calculated independently by ZUBAREV and KLIMOV. Their results correspond with those obtained in the present work.

BRAGINSKI S. I. and BUDKER G. I., *The physical phenomena in the initial stage of a discharge for incomplete ionization*, pp. 186–206 (completed 1952).

The present article is a preliminary report of the authors on the theoretical investigation of the initial stages of a high current discharge in hydrogen. This work was not completed, and in the present form contains only the setting up of the problem and an analysis of a series of individual questions.

The topics discussed are:

(1) The breakdown and the formation of a quasi-neutral plasma.
(2) The effect of the magnetic field of the current and the skin effect.
(3) Trapped radiation.

LEONTOVICH M. A. and SHAFRANOV V. D., *The stability of a flexible conductor in a longitudinal magnetic field*, pp. 207–213 (completed 1952).

The problem of the stability of non-elastic linear conductor of a circular cross-section carrying a current in a longitudinal magnetic field, is examined. It is well known that the field of the current produces instabilities with respect to the distortion of the form of the conductor. In the present note it is shown that if the external longitudinal field is sufficiently large, it leads to stability. The minimal value necessary for this field exceeds the value of the field of the current at the boundaries of the conductor. The problem is solved with the following assumptions: the conductivity is assumed to be infinite and the distortion of the shape of the conductor is not large. With these assumptions it is easy to find a field and a distribution of currents necessary for calculating the forces arising when the conductor is bent.

BUDKER G. I., *Electrical breakdown in a gas in the presence of a strong external magnetic field variable in time*, pp. 214–221 (completed 1952).

The magnetic field affects the breakdown for the most part in three respects:

(1) Decreases the transverse coefficient of diffusion.
(2) Causes a drift of electrons to the centre.
(3) Weakens the transverse conductivity so that the heating of electrons by the induced electric field can be neglected.

Breakdown is produced by the external longitudinal electric field. In a magnetic field, breakdown is

considerably easier than without the field, especially in the case when the magnetic field increases with time. A completely different dependence on gas density is found, which considerably facilitates the breakdown at low density. The discharge in an infinite straight tube and in a finite straight tube are examined. The problem of the discharge in a torus can be resolved into a problem in a straight tube with a sufficiently rapid growing magnetic field.

LEONTOVICH M. A., *The magnetic field of a linear current surrounded by an ideally conducting cylinder with a gap*, pp. 222–228 (completed 1952).

The presence of a gap decreases the force near it, which pulls the current toward the centre of the cylinder.

BRAGINSKI S. I., *Investigation of the axial region of a plasma pinch*, pp. 229–233 (completed 1952).

It is of interest to investigate what takes place near the axis of a pinch for a completely ionized plasma, where the ion and electron temperatures are equal. Close to the axis, where the magnetic field passes through zero, the trajectories of the particles have the form of snakes and not circles, as in the other parts of the pinch. Here, therefore, some peculiarities can be expected. The necessity for a special study of the region near the axis was first noted by BUDKER, who showed that at high temperature $\omega\tau$ may be large even inside the region of the snake-like trajectories. An attempt to investigate the region near the axis with the help of microscopic equations is being made.

KOMEL'KOV V. S. and SINITSIN V. I., *A piezo-electric method of investigating a powerful gas discharge*, pp. 234–242 (completed 1952).

A method is described for the registration of the form and magnitude of pressure pulses at the walls of the discharge chamber in the process of the development and passage in it of a gas discharge. This method permits the establishment of phase relationships between the discharge current and the pressures on the walls of the chamber and, making certain assumptions, to carry out an evaluation of the energy of the plasma particles. Barium titanate was used as a crystal.

BUDKER G. I., *Thermonuclear reaction in a potential hole of a negative charge*, pp. 243–248 (completed 1952).

The volume filled with electrons represents a potential hole for ions in which, generally speaking, there may be thermonuclear reactions. Unfortunately electromagnetic radiation of the electrons is considerably greater than the energy evolved by the reaction. Therefore, in general, such a system is not a reactor with a positive yield of energy.

ZUBAREV D. N. and KLIMOV V. N., *Stationary conditions of a magnetic thermonuclear reactor*, pp. 243–248 (completed 1952).

The volume filled with electrons represents a potential hole for ions in which, generally speaking, there may be thermonuclear reactions in the D-D and the D–T system.

This is a detailed article containing 11 Figs., many of them giving results of a large number of calculations. There are also 4 Tables, giving the sizes of the calculated nuclear discharges and their conditions of operation.

TRUBNIKOV V. A., *The instability of a plasma cylinder*, pp. 289–298 (completed 1952).

The stability conditions of a stationary cylindrical column of a completely ionized plasma with a longitudinal current uniformly distributed in cross-section, is examined. It is shown by a method of small perturbations that the axially symmetrical deformations having a periodic form along the axis of contraction, rise exponentially in orders of time $t \sim (a^{\chi}\, z/v)^{\frac{1}{2}}$ where a is the unperturbed radius of a cylinder, χ is the wave length of the perturbation, and v is the thermal velocity of the ions. This can lead to a complete disruption of the current in the circuit.

Note—Experimental data were obtained recently which qualitatively support the picture of the discharge described above. (Private communication of B. G. BREZHNEV.)

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BRAGINSKI S. I. and SHAFRANOV F. D., *The plasma pinch with heat loss at the electrodes*, pp. 3–19 (completed 1953).

An attempt is made to consider the plasma pinch with electrodes. The presence of electrodes leads to a variation in physical quantities along the axis, in contrast with what must take place in an infinite column as is considered in Vol. 1, p. 178, and BRAGINSKI (*J. exp. theor. Phys.* **33**, 459, 1957).

It is easy to show that in the case when the inertia and the viscosity of the plasma can be neglected, and when the electric field in the column is small with respect to the magnetic field, the pressure, the current

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density and the magnetic field in the pinch do not change along the axis.

BRAGINSKI S. I. and MIGDAL A. B., *The processes in a plasma column with high rate of increase of current*, pp. 20–25, (1951, and revised in 1956).

A short examination is made of the physical processes essential in a powerful pulse discharge; ionization, skin effect, motion of the ions toward the axis and their entrainment of the neutral gas, the convergent shock wave and cumulation on the axis, the compression of the plasma column taking into account the increase of the moving mass, temperature of the ions, and of the atoms.

The experimental investigations of a massive impulse discharge showed that with a rapid increase of the current in a cylindrical discharge chamber a plasma column is formed and compressed toward the axis. A qualitative theory of the compression has been developed by LEONTOVICH and OSOVETS (*Report of the Academy of Sciences*, 1953; *Atomic Energy* **1**, 81, 1956), on the basis of a hypothesis of the inertial character of this process. These results agree well with experiments. The present work is an attempt to make a qualitative investigation of the basic phenomena taking place in the process of formation of a plasma column, and to introduce some refinements of the physical picture of its compression.

BRAGINSKI S. I. and SHAFRANOV V. D., *The plasma pinch in the presence of a longitudinal magnetic field*, pp. 26–80 (completed 1953).

An investigation is given of the plasma column in the presence of an internal longitudinal field. Preliminary presentation is given of the basic results referring to a pinch without a longitudinal field.

The following are the headings of this long article:

(1) Equations describing the pinch.
(2) The relation of the pinch to the temperature.
(3) Rapid and slow processes.
(4) Heat balance equations.
(5) Change of the radius of the pinch with time.
(6) Notes concerning the heating of the plasma and the compression of the pinch.
(7) A pinch with heat losses.
(8) Similarity regimes.
(9) Conditions for non-rupture from the walls.
(10) System of equations in the presence of a longitudinal field.
(11) Stationary case.
(12) Rapid processes.
(13) Similarity solutions.
(14) A pinch produced by a longitudinal field.

ARTSIMOVICH L. A., *The passing of large currents through a plasma in the presence of a longitudinal magnetic field*, pp. 81–86 (completed 1953).

The question is examined of the behaviour of a plasma pinch produced by rather large currents passing through a plasma, in the case when the whole system is in a constant field, or almost constant field. It is assumed that one can neglect the gas kinetic pressure of the plasma (rapid pulse process). The conductivity of the plasma is considered infinite.

ARTSIMOVICH L. A., *Magnetic current in a compressing cylinder*, pp. 87–100 (completed 1953).

Processes are examined in a compressed conducting cylinder present in a magnetic field. Formulae are given for several cases of the change of the cylinder radius as a function of time. In addition to this, an analysis is given of the experimental data and the possibility of determining the conductivity of the cylinder.

ARTSIMOVICH L. A., *Analysis of the equations of the compression of the pinch in the presence of an external magnetic field*, pp. 101–108 (completed 1954).

An analysis is made of the equations describing the compression of a gaseous column in the presence of a constant longitudinal magnetic field. The case is examined when the 'magnetic time' is small compared to the time of the current change, and approximate equations are determined for the radius of the pinch.

KOGAN V. I., *The yield of thermonuclear reactions*, pp. 109–129 (completed 1953).

The article investigates several questions dealing with the yield of the D–D reaction, and in particular:

(1) The region of the applicability of the simple analytical expression for the yield (the Bethe-Atkinson formula).
(2) The influence of the 'cutting of the tails' of the Maxwellian distribution of ion velocities.
(3) For a given energy introduced into the system, the optimum number of particles for a polytropic law of compression of the reacting plasma.
(4) For a given current the optimum linear number of particles in a system with a plasma current working with a similarity regime condition (taking into account a temperature difference between the ions and the electrons).

SHAFRANOV V. D., *Stability of a plasma pinch in the presence of a longitudinal magnetic field and a conducting envelope*, pp. 130–143 (The work was carried out in 1953 and revised subsequently).

The basic results were presented in a compressed form previously (*Atomic Energy* **5**, 38, 1956). Analogous questions were discussed by KRUSKAL and SCHWARZSCHILD (*Proc. Roy. Soc.* **A223**, 348, 1954); TAYLER (*Proc. Phys. Soc.* **B70**, 31, 1957); TAYLER (*Proc. Phys. Soc.* **B70**, 1049, 1957); ROSENBLUTH (*Proc. 3rd Int. Conf. on Ionization Phenomena in Gases, Venice*, 1957).

Investigation of the stability of a plasma pinch in the presence of a longitudinal magnetic field, shows that there are two possible methods of stabilization of the column with the help of a longitudinal magnetic field;

(1) Stability by means of internal frozen in longitudinal field greater than the external field in the presence of an envelope. In that case, there is a region of stability with respect to arbitrary perturbations.
(2) Stability in the presence of a strong external longitudinal field. In that case, a calculation made in the linear approximation, shows that not all forms of perturbation can be stabilized. It is possible, however, to stabilize the more dangerous of the perturbations, in which the axis of the pinch is displaced from its equilibrium position.

VOLKOV T. F., *The stability of a plasma cylinder in an external magnetic field*, pp. 144–149 (completed 1953).

This article investigates the question of the stability with respect to 'sausage' type perturbations of the plasma cylinder with a current in an external field directed along the axis. It is assumed that the current is distributed uniformly in cross-section. It is shown that for a sufficiently large magnetic field the perturbations of the type examined are stable.

BORZUNOV N. A. and ORLINSKI D. V., *Distribution of the intensity of the neutron radiation along the axis of a straight tube for a high current pulsed discharge in deuterium*, pp. 150–164 (completed 1954).

Conclusions:

(1) The intensity of the neutron radiation of a high current pulsed discharge in deuterium increases linearly with increase of the initial potential (from 30 to 50 kV). In addition to this, the neutron radiation of the discharge exceeds the cosmic background in the range of $N = 1{\cdot}5 \times 10^{17}$ to $3{\cdot}5 \times 10^{18}$ atoms per cm of length.
(2) In determinining the distribution curve one does not obtain reproducible results. This is evidence of the statistical character of the phenomena taking part in the discharge. Results of investigation of the experiments on a tube of larger bore (with a distance of the counters to the axis of 20·5 cm) give a basis for the conclusion that the neutron radiation is not distributed uniformly along the length but is concentrated in definite, rather small regions which may be several in number. In other words, the neutron radiation in the discharge has a local character and is concentrated in a region of the order of the diameter of the discharge tube.

OSOVETS S. M., *Mechanism of the observed neutron emission*, pp. 165–169 (completed 1954).

In the process of investigating the phenomena which arise in powerful pulsed discharges in deuterium, it was found that under certain conditions there is a noticeable neutron emission which is accompanied usually by the emission of hard X-ray quanta. (ARTSIMOVICH and ANDRIANOV, *Atomic Energy* **3**, 1956; LUK'YANOV and PODGORNI, *Atomic Energy* **3**, 1956).

It is shown that for the D–D reaction, the observed intensities may take place only because of interactions of deuterons with energies many times greater than the temperature of the plasma and cannot be explained by a thermonuclear reaction.

KOMEL'KOV V. S., MOROZOVA T. I. and SKVORTSOV YU. V., *Investigation of a powerful discharge in deuterium*, pp. 170–184 (completed 1954).

Oscillographs I and V are given for a discharge in deuterium for $C = 24$ to $36\ \mu F$, $I_{max} = 650$ kA, diameter of the chamber 630 mm, for the pressure range of 0·0025 to 0·1 mm Hg, and for initial potentials $V_0 = 42$ to 100 kV. Several peculiarities were observed in the current. The integrated yield of neutrons was registered in its dependence on C, V_0, P_0. The time of appearance, and the length of the neutron pulses was determined. An investigation was made of the X-radiation accompanying the neutron emission.

This article contains experimental data, a diagram with a scale of the apparatus, the electronic characteristics of the circuit and 20 Figs. in all.

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ANDRIANOV A. N., BRAZILEVSKAYA O. A. and PROKHROV YU. G., *Investigations of a pulsed discharge in gases for a current of* 500 *kA*, pp. 185–211 (completed 1956).

Results are presented on the measurement of the distribution of the magnetic and electric fields within a powerful gas discharge. These measurements are part of a large programme of work for investigating high current pulsed discharges in gases. (cf. *Atomnaya Energiya* **3,** 1956). The basic goal was the investigation of discharges in deuterium, but in addition to this, we have also investigated discharges in other gases (hydrogen, helium, argon, xenon)

The sub-headings of this article are:

(1) Description of the apparatus and the installation.
(2) General characteristics of a powerful gas discharge.
(3) Distribution of the magnetic field and the current density in the cross-section of the discharge.
(4) Longitudinal electric fields in the gas discharge.

KIRILLOV V. D., *Measurement of plasma conductivity for currents of long duration*, pp. 212–225 (completed 1954).

The work describes the methods and experimental results of investigating a powerful low frequency discharge in deuterium (currents up to 95 kA at a frequency of 270 c/s).

Oscillograms are given of the voltage for discharges with a longitudinal magnetic field and without it, for curves of the change of the inter-electrode resistance as a function of time, high-speed photographs for the potential at the condensers equal to 6 kV, and the curves for the distribution of the potential along the axis of the discharge tube.

The experimental results with a low frequency gas discharge showed that:

(1) The discharge begins in the centre of the tube.
(2) The plasma pinch retaining its stability expands toward the walls. During the time of expansion the total resistance of the gas discharge drops.
(3) At the end of the 'plateau', which corresponds to the expansion, the conductivity stops increasing, reaching the value of 5 to 15 $\times$ 10^{13} CGS units independent of the conditions.
(4) With increase of dI/dt, the length of the plateau decreases.
(5) The potential of the electric field is strongly inhomogeneous along the axis of the chamber.

BABIKOV V. V., *The theory of bremsstrahlung for non-relativistic radiation*, pp. 226–237 (completed 1954).

Corrections were obtained to the important members of the Born and Quasi-classical approximation for the spectral distribution of the intensity of the bremsstrahlung of electrons in a Coulomb field.

Note—Questions analogous to those investigated above were examined in the following recent work: KUMMERER (*Z. Phys.* **147,** 373, 1957); GUGGENBERGER (*Z. Phys.* **149,** 523, 1957).

OSOVETS S. M., *A plasma discharge in an electromagnetic field*, pp. 238–241 (completed 1953).

The basis of the theory of an electrode-less gas discharge of a definite configuration is presented. In a discharge of this type, conditions may be realized up to a certain degree analogous to those which take place in the acceleration of electrons in a vacuum betatron. Just as in the betatron there is a possibility of increasing the electron energy, so in the gas discharge considered here the plasma may be heated.

OSOVETS S. M., PETROV YU. F. and SCHEDRIN N. I., *Investigation of a gas discharge in a simply connected region*, pp. 242–263 (completed 1955).

Results are given of the investigation of a gas discharge in a simply connected region arising under the action of a variable magnetic field. It is shown that in this case a plasma pinch is formed which is compressed to the centre of the discharge vessel. The theoretical investigations show that the choice of a special form of a magnetic field may give a region in which such a plasma pinch is in a state of equilibrium. The investigations were carried out on a system with a magnetic field satisfying these conditions. In this system, hard X-radiation was observed (several hundred keV) indicating the fundamental possibility of obtaining in installations of this type a betatron acceleration of electrons without preliminary injection.

NASEDKIN YU. F., *Investigation of a ring gas discharge in a transverse magnetic field*, pp. 264–282 (completed 1955).

With the purpose of decreasing the influence of the active resistance of a gas pinch on the course of the discharge, an installation presented in this work was constructed. It insures the necessary form of the magnetic field and has a small inductance equal to 0·15 μH and a large coefficient of coupling ($K = 0{\cdot}35$). The distribution of the secondary current in the volume of the chamber was studied.

The results of the experiments give a basis for considering that in the course of the first half period, the secondary current is separated from the walls and situated near the calculated stable orbit. In the region of the maximum primary current, the secondary current is separated from the orbit and moves toward the centre as a unit. In the second and subsequent half-periods, the discharge fills the whole volume.

BELYEV C. T. and BUDKER G. I., *Relativistic plasma in variable fields*, pp. 283–329 (completed 1953).

An investigation is made of the behaviour of a relativistic gas of charged particles present in a strong constant-in-time and, generally speaking, non-homogeneous magnetic and electric field with an exciting variable electromagnetic field, the frequency of which is much greater than the frequency of collisions, so that one can neglect the latter.

BUDKER G. I. and BELYEV S. T., *Boltzmann equation for an electron gas for rare collisions*, pp. 330–354 (completed 1954).

A method of simplifying kinetic equations for rare collisions is proposed. A corresponding equation is found for a relativistic gas which can be used for the investigation of a stationary state of a stabilized electron beam.

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BUDKER G. I., *Thermonuclear reactions in a system with magnetic mirrors. The problem of direct transformation of thermonuclear energy into electrical energy*, pp. 3–31 (completed 1954).

The physical principles and preliminary calculations are presented in this investigation concerning the methods proposed by the author for realizing the thermonuclear reaction in a straight cylinder closed at the ends with magnetic mirrors. A method is investigated for igniting the discharge in a high vacuum and for heating the plasma with the help of a magnetic piston. An examination is also made of the problem of direct transformation of nuclear energy into electrical energy.

BUDKER G. L., *Several problems of the spatial stability of a ring current in plasma*, pp. 32–40 (completed 1951).

This investigation considers several problems associated with the stability of the ring current in plasma. The stability is considered only with respect to slow changes, during which the equilibrium values of temperature and density are reached. Rapid changes and those occurring in times determined by inertia are not investigated.

BELYEEV S. T. and BUDKER G. L., *Multi-quanta recombination in an ionized gas*, pp. 41–49 (completed 1955).

With the help of the classical Boltzmann equation the process of electron transition to the ground level of the atom, due to multiple collisions and emission of small quanta has been investigated. A method of simplifying the Boltzmann equation when collisions are infrequent is used.

BELYEEV S. T., *The Boltzmann equation for rarefied gases in strong fields*, pp. 50–65 (completed 1955).

A method is developed for obtaining approximate kinetic equations for dilute ionized gases in strong external fields. The small parameter in this approximation is the ratio of the period of characteristic vibrations of the particles in an external field to the time of the mean free path. The approximate equation has the form of the Fokker-Planck equation with a smaller number of variables.

BELYEEV S. T., *The kinetics of an ionized gas in a strong magnetic field*, pp. 66–85 (completed 1955).

An ionized gas in a strong magnetic field is considered as an assembly of Larmor rings. The Boltzmann equation is obtained for the distribution function of the Larmor rings. Its solution is examined in the hydrodynamical approximation. The coefficient of the electron-ion diffusion is calculated for a uniform magnetic field. In an inhomogeneous field fluxes are found, neglecting collisions.

BABIKOV V. V. and KOGAN V. I., *Radiative heat transfer from a dense high temperature plasma*, pp. 86–98 (completed 1955).

Because of its application to the problem of heating dense or condensed media, a calculation is made of the heat loss of a layer of plasma by bremsstrahlung for any opacity.

In the present article, the question of radiative opacity and radiative heat losses of a dense high temperature plasma is investigated for those values of density, temperature and size when, in general, one or the other of two limiting cases is not applicable: (1) the free emergence of bremsstrahlung from the whole volume of the plasma, (2) black radiation from the plasma surface, the radiation being effectively

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trapped within it. Investigation of this question is important for the problem of heating dense or condensed media to temperatures at which thermonuclear reactions can be brought about. The reason is that, firstly, heat transfer by the bremsstrahlung mechanism is, for a wide range of parameters, the main contribution to the total heat transfer from the plasma; secondly, the transition from the transparent medium to the 'black' medium occurs over a considerable part of this range.

Since under the conditions examined (density of the plasma 10^{19} to 10^{23} ion/cm^3, and temperature 10 eV to 1 keV, the plasma is practically completely ionized, the basic mechanism of the absorption of radiation, due to the opacity of the plasma, is the inverse bremsstrahlung (photo-effect on free electrons).

KOGAN V. I., *Recombination radiation from a hydrogen plasma*, pp. 99–103 (completed 1956).

A calculation is made of the total intensity of recombination radiation of a rarefied hydrogen plasma as a function of the electron temperature (taking into account recombination at excited levels). A comparison is made of this quantity with the intensity of bremsstrahlung.

The basic mechanism of radiant heat loss from a high temperature hydrogen plasma ($T \geqslant 100$ eV), as is well known is electron bremsstrahlung. On the other hand, for sufficiently low electron temperatures (of the order of several electron volts) the dominant contribution to the total radiation is given by the processes of recombination of electrons with ions. Therefore, for comparatively low temperatures (less than tens of electron volts) it is of interest to calculate the intensity of the recombination radiation and to compare it with the intensity of bremsstrahlung.

It is shown that the intensity of the recombination radiation decreases linearly as a function of the electron temperature, while the radiation due to bremsstrahlung increases with electron temperature, giving for the total losses a minimum at a temperature of about 30 eV.

TRUBNIKOV B. A., *Electromagnetic waves in a relativistic plasma in a magnetic field*, pp. 104–113 (completed 1956).

The relativistic tensor for the dielectric susceptibility of a plasma in a homogeneous magnetic field is derived. Thermal motion of the electrons is described by a Maxwellian (relativistic) distribution. Collisions are considered rare. With the help of the dispersion equation, propagation of waves of the form exp $[i(kr - \omega t)]$ is investigated. For frequencies $\omega > (eH)/(mc)$ the emissive power of the medium and the absorption coefficients of 'ordinary' and 'extraordinary' waves are calculated.

The dielectric properties of a relativistic plasma in a magnetic field is first discussed in the work of GALITSKI and MIGDAL (Vol. 1, p. 161). Using a quantum mechanical approach, the authors found for the tensor an expression having cumbersome series. In the present investigation this same problem is examined classically and a rather simple formula is obtained for the tensor.

KUDRYEVTSEV V. S., *The electron distribution function in a plasma in a magnetic field*, pp. 114–120 (completed 1956).

A distribution function for electrons is calculated for a two-temperature plasma for the case when the electrons obtain energy by collisions from the ions at high temperature and emit it by their motion in a magnetic field. The stationary process is examined and the plasma is considered completely transparent.

TRUBNIKOV B. A. and BAZHANOVA A. E., *Magnetic radiation from a plasma layer*, pp. 121–147 (completed 1957).

In this work calculation is made of the magnetic bremsstrahlung from a plane layer of plasma in a magnetic field parallel to the boundaries of the layer. The energy of the electrons is just relativistic. Calculations are carried out for two forms of energy distribution of electrons—Maxwellian and a cut-off Maxwellian distribution. It is shown that the radiation is trapped to the extent necessary for a self-sustaining thermonuclear D–D reaction only when the dimensions of the layer are considerable, lying at the limit of achievability.

VOLKOV T. F., *The decay of an arbitrary discontinuity in a continuous medium*, pp. 148–152 (completed 1956).

A particular case of the decay of an arbitrary discontinuity in a continuous medium is discussed in order to estimate the temperature which can be obtained by the collision of dense beams of particles with one another or with a solid target. Some brief considerations on the possible effect of a magnetic field also are given.

The treatment is an extreme idealization of the conditions found in laboratory equipment, but it

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may be realized in collisions of gaseous masses of cosmic dimensions.

RUDAKOV L. I. and SAGDEEV R. Z., *High-frequency heating of a plasma*, pp. 153–164 (completed 1956).

A method of heating plasma ions is considered, based on the utilization of cyclotron resonance. When the fields in the plasma due to the motion of ions and electrons are allowed for, the motion of the particles may be considerably modified, and the heating mechanism may no longer operate. To find the conditions under which ions may be accelerated, the problem of particle motion in an external field is considered in Part 1, without allowance for the self-consistent field of the plasma itself. The effect of this field is analysed in Part 2, using the hydrodynamical approximation.

OSOVETS S. M., *The theory of rapid processes*, pp. 165–181 (completed 1957).

Calculations are given for the rapid compression of a plasma column, taking into account the varying inductance of the circuit. The basic relations for the temperature of the compressed column and the compression time are derived. The conditions for separation of the plasma from the walls of the vessel are considered, and the mean temperature of the gas at the moment of separation is calculated.

NASEDKIN YU. F. and OSOVETS S. M., *Investigation of a powerful ring gas discharge when there is an equilibrium orbit*, pp. 182–195 (completed 1956).

A powerful ring gas discharge in a variable magnetic field is considered, the field having a special configuration which ensures the existence of a stable equilibrium current orbit. The current distribution over the volume of the vacuum chamber is investigated. The maximum energy of motion of the ring towards the centre of the chamber after breaking away from the orbit is estimated to be $\sim$1500 eV. The residual magnetic flux and resistance of the gas ring are estimated.

OSOVETS S. M. and SCHEDRIN N. I., *A plasma ring with a resistance*, pp. 196–213 (completed 1957).

The first part of the paper gives a theoretical discussion of the conditions of equilibrium in the process of contraction of a plasma ring of finite conductivity. It is shown that, if the ring is to contract into a ball, the residual magnetic flux must be balanced by some field constant in time. In the second part, results of experiments with a compensating field are given, confirming the conclusions reached in the first part. The conditions under which the residual flux is completely balanced, and the ring contracts to a ball are derived.

Experiments have shown that, when the gas ring contracts to the centre, about 1/4 of the internal magnetic flux is frozen inside it. This residual field prevents the contraction of the orbit to small dimensions. In addition, part of the magnetic energy is spent on the building up of the field. If additional central field is used, a compensation of the residual flux is observed when this field reaches a certain value. In this case, it would be expected that the magnetic energy mainly is converted into kinetic energy of directed motion of the particles to the centre, with the formation of a dense 'ball'.

NASEDKIN YU. F. and PAVLOV E. I., *The effect of the shape of the magnetic field on a ring gas discharge*, pp. 214–230 (completed 1957).

A study is made of the behaviour of a gas current ring in a betatron field and the distribution of current density over the cross-section of the chamber. It is found that the shape of the field has an influence on the behaviour of a current ring. For low gas pressures in the chamber, X-ray emission is detected. The experiments were carried out with deuterium and argon.

FILIPPOV N. V., *Investigation of pressures in a powerful pulsed gas discharge using a piezo-electric measuring device*, pp. 231–249 (completed 1956).

A description is given of the construction of a piezo-electric instrument for measuring pressures in a pulsed discharge and the method of using it in a cylindrical discharge chamber at currents from 300 to 500 kA. Results of investigations of the pressure near the centre of the chamber are given for a discharge in deuterium of a condenser bank of 62 μF capacity. A new method is described for calibrating this device directly in the discharge chamber using a diaphragm inertial device.

ANTROPOV G. M., BELYEVV V. A. and ROMANOVSKI M. K., *The behaviour of fast electrons in the electron model of a trap with magnetic mirrors*, pp. 250–258 (completed 1957).

One of the methods of creating a hot plasma can be the accumulation of fast ions in a volume with a magnetic field of a special form in which the particles

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are kept for a long time from hitting the walls. In such magnetic traps, regions with large values of H play the role of mirrors, from which the charged particles are reflected corresponding to the invariance of the quantity V^2/H in a magnetic field, which changes only slightly over the Larmor radius of a particle. The simplest form of the magnetic trap is the field of a straight solenoid, which is increased at the ends. This form was investigated using an electron model although this is not the only one, nor possibly the best configuration of the magnetic field of the trap.

Conclusions:

(1) The injected electrons moving in the magnetic field of the trap make up to a thousand oscillations between the mirrors.

(2) The lifetime of fast electrons in the system is determined not by the constancy of the adiabatic invariant, but by the interaction of the electrons with the residual gas and their return to the source.

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FIRSOV O. B., *The repulsion of charged particles from regions of strong magnetic fields*, pp. 259–267 (completed 1956).

This work attempts to determine the validity of the adiabatic invariant for the conditions found in magnetic traps with axially symmetric field.

RUDAKOV L. I. and SAGDEEV R. Z., *A quasi-hydrodynamic description of a rarefied plasma in a magnetic field*, pp. 268–277 (completed 1957).

The Boltzmann equation which describes the motion of the ions of a plasma in a magnetic field in the absence of collisions and averaged over the Larmor revolution, is generalized for the non-static case. The system of equations for the lower moments of the distribution function of the electrons and the ions is analogous to the equations of two fluid hydrodynamics. While formally the mean free path is infinite, it is replaced by the Larmor radius of the ion or electron. Using the equations obtained, the wave solutions corresponding to magnetohydrodynamic and acoustical oscillations have been investigated. For a sufficiently large anisotropy of the pressure tensor instabilities arise which lead to the destruction of the homogeneity of the plasma. The study of the motion of the completely ionized plasma in strong electromagnetic fields taking into account the field of the plasma and using the Boltzmann equation is difficult. The usual hydrodynamical discussion is valid only for high densities and low temperatures, when the mean free path between collisions is much less than the characteristic dimension. Recently CHEW *et al.* (*Proc. Roy. Soc.* **236,** 112, 1956) have investigated the case of a rarefied plasma in a magnetic field neglecting collisions. They showed that the expansion of the Boltzmann equation in powers of M/e (M being the mass of the particle and e the charge) leads to a closed system of equations for the lower moments of the distribution function formally analogous to the equations of magnetohydrodynamics. However, this is valid only for the motion of plasma across the magnetic field. This similarity of results obtained in two apparently opposite cases is explained by the fact that the magnetic field 'burns the ions' and so makes symmetrical the distribution of their velocities in a plane perpendicular to the magnetic field. In this sense the action of the magnetic field is analogous to that of collisions. In this note we are deducing the equations of hydrodynamics of two fluids (ions and electrons) starting from the Boltzmann equation describing the averaging of the motion of the ions or electrons over the Larmor revolution.

VEDENOV A. A. and SAGDEEV R. Z., *Some properties of a plasma with anisotropic ion velocity distribution in a magnetic field*, pp. 278–284 (completed 1957).

The possibility of instability of a plasma in a magnetic field owing to anisotropy of the ion velocity distribution is considered.

In the usual gas with non-Maxwellian velocity distribution, equilibrium is established in time, t, of order of magnitude of that between collisions. In a plasma the long-range forces may cause collective motion which develops in time smaller than t (example of such collective motion is the formation of Langmuir oscillations in the plasma with non-Maxwellian electron velocity distribution (BOHM and GROSS, *Phys. Rev.* **75,** 1864, 1949).

The influence of the external magnetic field on these effects was studied by GROSS (*Phys. Rev.* **82,** 232, 1951) using the Boltzmann equation.

Investigating the electron oscillations of a plasma, GROSS naturally neglected the magnetic field of the wave as being of order v/c. The situation changes substantially if the ion (magnetohydrodynamic) oscillations are under discussion. The electric fields in low frequency oscillations are largely screened by the motion of the electrons, and magnetic effects may be of importance. In the article by RUDAKOV and SAGDEEV (this issue, p. 268) from equations for the moments of the distribution function in the drift approximation, criteria were obtained for the

formation of specific instabilities when the longitudinal and transverse temperatures are markedly different. The discussion corresponds to a hydrodynamic derivation of the dispersion equation. In the present work this problem is investigated using the Boltzmann equation, and criteria are found for instability neglecting collisions. An allowance for the magnetic field of the wave for a sufficiently anisotropic distribution function leads to a transfer of energy from transverse to longitudinal motion (or vice-versa) by the excitation of collective motions associated with ion oscillations of the plasma.

KADOMTSEV B. B., *Magnetic traps with a 'corrugated' field*, pp. 285–299 (completed 1956).

Two simple traps with a 'corrugated' magnetic field are described, whose mode of operation is based on the increased drift of particles round the axis of the system brought about by an additional periodic field. The first of these operates similarly to a simple trap with magnetic mirrors, i.e. the exit of particles from it is determined by Coulomb collisions. In the second case the loss of particles arises from a toroidal drift, and in this sense it resembles a torus. There are probably better traps of this type and for this reason it would be of interest to study in more detail the motion of particles in magnetic traps which, one would hope, would make possible the choice of the best configuration. Great interest is shown also in the study of the behaviour in such traps, not only of individual particles, but also of plasma, taking into account its electric and magnetic fields and other complicating factors. In particular the stability of such systems should be investigated.

BRAGINSKI S. I. and KADOMTSEV B. B., *Stabilization of a plasma by the use of guard conductors*, pp. 300–326 (completed 1957).

A method is proposed for stabilizing the boundary of a plasma, whose pressure is balanced by a magnetic field. Stability is achieved by means of metallic guard conductors. They form a network on the boundary of the plasma, and carry part of the pressure of the magnetic field. This excess pressure prevents the plasma from passing outside the network. Such a system is shown to be stable for small oscillations. The shape of the plasma boundary near the network lines is considered.

FIRSOV O. B., *A plasma in a magnetic grid*, pp. 327–335 (completed 1957).

The flow of an electron-ion plasma along lines of magnetic force surrounded by a constant magnetic field is considered. The magnetic field configuration is such that there is no field inside the plasma. The prospects of using such a system in constructing a thermonuclear reactor are assessed.

VOLKOV T. F., *A stationary plasma density distribution in an electromagnetic field*, pp. 336–345 (Work done in 1957, and revised in 1958).

Using the hydrodynamical approximation, the problem of the stationary self-consistent density distribution of a plasma in a standing electromagnetic wave is solved. It is shown that, in this approximation, there is no exact solution for which the plasma density is zero in some region of space. The effect of the magnetic field and of the walls bounding the plasma also is considered.

SAGDEEV, R. Z., *Containment of a plasma by the pressure of a standing electromagnetic wave*, pp. 346–361 (completed 1957).

Recently it has been of general interest to consider the problem of containing a high temperature plasma in a high-frequency electromagnetic field. The possibility of such a containment rests on the fact that an alternating electromagnetic field does not penetrate into a conductor and thereby causes a pressure drop at the boundary of the conductor. This principle has long been used in the well-known experiments for counteracting the forces of gravity by the ponderomotive forces of eddy currents arising in the skin layer of solid conductors. In the present paper some of the problems are investigated concerning the equilibrium and the stability of a heated plasma contained by the pressure of the alternating electromagnetic field.

In many cases the plasma can be described with a sufficient degree of accuracy by the equations of gas dynamics. The criterion for the validity of such an approach is the small magnitude of the mean free path of the ion or electron compared with the characteristic dimension L. The Coulomb collision cross-section decreases with increase of the relative energy of the colliding particles as the inverse square of the energy. This has a decisive effect on the criterion of applicability of the equations of gas dynamics. In the cases of practical interest the range of temperature corresponding to a strongly ionized plasma lies outside of the regions of the hydrodynamic approximation. In the presence of the constant magnetic field this criterion, however, can be replaced by another one, at least for the description of the motion of the plasma across the magnetic field, namely that the Larmor

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radius r_H of the ion or electron should be small compared with L.

In the general case the plasma must be described by the Boltzmann equation for each type of ion. This approach, however, is extremely complex and scarcely allows the derivation of any specific results. A good approximation to the description of the high temperature plasma may be to consider the Boltzmann equation without the collision integral, considering later the collisions as a small correction. In the present article the case of a plasma is investigated which has such high ion and electron temperature that collisions can be neglected. As a basis for the calculation of the equilibrium state, a method using the Boltzmann equation without the collision term, but including the self-consistent electromagnetic field is employed.

Finally, the equations of hydrodynamics are used to investigate the stability of the boundary between the heated plasma and the standing electromagnetic wave.

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OSOVETS S. M., *Containment of a plasma by a travelling magnetic field*, pp. 3–15 (completed 1957).

The problem of containing a hot plasma by means of a field of a travelling wave is discussed. Expressions are obtained for magnetic fields and magnetic pressures produced on the surface of the plasma by a system of currents displaced in phase with respect to one another. Conditions are found under which the perturbation of the surface of the plasma can be considered small. Starting from these conditions, relations are found which determine the basic parameters of the circuit which produces the travelling field. Relations are derived characterizing the energy balance of a thermonuclear reactor in which the plasma is contained by a travelling field.

KADOMTSEV B. B., *Hydrodynamics of a low pressure plasma*, pp. 16–23 (completed 1957).

Approximate equations have been obtained of the motion of a plasma, the pressure of which is small compared with that of the magnetic field. The problem of the equilibrium and stability of such a plasma is discussed.

BRAGINSKI S. I. and KAZANTSEV A. P., *Magnetohydrodynamic waves in a rarefied plasma*, pp. 24–31 (completed 1958).

An investigation is made of the magnetohydrodynamic wave with arbitrary direction of propagation starting from the kinetic equations with a self-consistent field neglecting collisions. As in a conducting liquid, in a rarefield plasma three types of magnetohydrodynamic waves can be considered. It is shown that in the absence of collisions these waves undergo a characteristic decay.

SOLV'EV L. S., *The motion of charged particles in a magnetic trap*, pp. 32–41 (completed 1957).

In the approximation of the drift theory a calculation is made of the motion of a particle in the field of a trap with magnetic mirrors. The length of the trajectory of the particle, the tolerance for the curvature of the axis of the trap, and the influence of supplementary gaps of the field on the axis are determined. For a given parabolic approximation of the field on the axis it is possible to integrate in a closed form the equations of motion of the particle.

VEDENOV A. A. and RUDAKOV L. I., *The motion of a charged particle in a rapidly alternating electromagnetic field*, pp. 42–48 (completed 1958).

The motion of particles in magnetic and electric fields slowly varying in time and space, has been investigated. The solution is obtained in the form of a series in increasing powers of small parameters Ω/ω r_H/L (Ω is the frequency of oscillation of the electromagnetic field, ω is the Larmor frequency, r the Larmor radius, L is the characteristic distance of the change of the field in space).

In the present investigation it is shown in one case how the limitation $\Omega/\omega \ll 1$ can be removed making use of the small magnitude of r/L and δ/L, where δ is the displacement of the particle in time $1/\Omega$.

ORLINSKI D. V., *Stabilization of a plasma using a system of rods*, pp. 49–53 (completed 1958).

S. I. BRAGINSKI and B. B. KADOMTSEV (vol. 3, p. 300) proposed using transverse rods carrying current for stabilizing a plasma. In the present note results of preliminary experiments to study the influence of transverse rods on the development of a pulsed discharge in a cylindrical chamber are presented.

RUDAKOV L. I. and SAGDEEV R. Z., *Investigation of the stability of a cylindrical plasma column using the Boltzmann equation*, pp. 54–60 (completed 1958).

An investigation was made of the stability of a cylindrical plasma current column in a longitudinal magnetic field. The method of investigation was to

use the Boltzmann equation in a 'drift' approximation derived in the previous article, (Vol. 3, p. 268) neglecting collisions. In the case of isotropy of the unperturbed pressure, the criterion coincides with that obtained in the magnetohydrodynamic approximation. In the case of the inequality of the longitudinal and transverse components of pressure, there is a possibility of a new instability even in the absence of a current along the axis of the column.

SHAFRANOV V. D., *Stability of a plasma column with a distributed current*, pp. 61–69 (completed 1958).

The first investigations of the stability of the plasma column in the presence of a longitudinal magnetic field (Vol. I, p. 207, and Vol. 2, p. 144) were based on the assumption that the whole current was concentrated in an infinitely small surface layer. It was shown in this case that if the internal longitudinal field was greater than the outer field, then the column could be stable in the presence of a shell of high conductivity independently of the ratio of the length of the column to its radius. In the reverse case the column could be of a limited length L. For the case when the internal field is equal to the external field, the condition of stability has the form $L\,(2\pi\alpha\, h = cH_z/(I/\pi\alpha^2))$. It is of some interest to find how these conditions are changed in the presence of volume currents. In the work of VOLKOV (Vol. 2, p. 144) it is shown that the criterion of stability for an axially symmetric perturbation for a uniform axial current does not differ markedly from the stability criteria obtained for the case of surface currents.

Usually it is considered more difficult to obtain criteria of stability with respect to the bending of the column and more complicated perturbations. For a uniform current, it was found possible to investigate rather simply all these perturbations.

POLIEVKTOV-NIKOLADZE N. N., *Calculation of quasi-stationary electromagnetic field in a system with toroidal geometry*, pp. 70–80 (completed 1958).

The analytical calculations of non-stationary electromagnetic fields in toroidal systems is extraordinarily difficult because Maxwell's equations in this case do not permit the separation of variables. It is shown that for thin conductors of toroidal geometry, the above difficulty disappears and the problem resolves itself in the solution, generally speaking, of a system of integral equations for functions with one independent variable, which can be solved fairly simply by the usual numerical methods.

DEMICHEV V. F. and PROKHOROV YU. G., *Investigation of the neutron emission arising in a gaseous discharge with a current of* 160 *kA*, pp. 81–86 (completed 1957).

An investigation is made of the neutron emission from a discharge in a cylindrical tube with a current 160 kA. It is shown that the neutron emission is due to the presence in the discharge of deuterons accelerated to high energies.

From the table it can be seen that the neutron radiation is non-isotropic. Experimental data obtained do not support the assumption of the localization of the source of the neutrons in a small region of a discharge or of uniform distribution of the intensity along the axis. The data are consistent only with the assumption that the intensity of the source of neutrons increases along the axis from the anode to the cathode. A qualitative evaluation shows that it must change according to x^{Ω}, where x is the distance along the axis from the anode and Ω lies between 1 and 2. This is consistent with the assumption that the neutron emission arises from reactions caused by accelerated deuterons.

TRUBNIKOV B. A., *A possible mechanism for the neutron emission from powerful pulsed discharges in deuterium*, pp. 87–97 (completed 1958).

The mechanism is based on the assumption that in the process of the development of perturbations of the plasma column, the plasma may be completely pushed out by the pressure of the magnetic field from several regions of the column. In such perturbations strong electrical field may arise which may cause the acceleration of deuterium ions to produce nuclear reactions.

There is a reference to the paper of ANDERSON *et al.* (*Venice Conference*, 1957) and a discussion of the COLGATE mechanism. The following remarks are made about it, in addition to others:

Thus, in contrast to COLGATE's work where the mechanism of acceleration in fields is assumed to take place before the moment of the formation of the perturbation, we are considering acceleration in fields which arise after the formation of perturbations. For the formation of sufficiently strong fields in this second mechanism we need simpler conditions. In particular, we do not have to assume that the radius of the conducting channel decreases to the size of the order of tenths of millimeter as is done in COLGATE's work. Also, it turns out that for the evaluation of the magnitude of the electric fields we do not have to know in general, the microscopic picture of what occurs during the formation of the perturbation.

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Therefore, the whole description can be carried out phenomenologically.

VOLKOV T. F., *The influence of a high frequency electromagnetic field on plasma oscillations*, pp. 98–108 (completed 1958).

In this article the hydrodynamic theory of plasma oscillations in the field of a travelling electromagnetic wave is examined. It is shown that in this case the frequencies of the acoustical and Langmuir oscillations become functions of the amplitude of the high frequency field. The possible mechanisms for the formation of instabilities are discussed.

The conditions of the problem demand that the frequency of the electromagnetic field is much greater than the frequency of Langmuir oscillations. This is realized in practice only in the case of a sufficiently rarefied plasma. In conclusions, it is noted that the hydrodynamic solution of the problem presented cannot be considered exact because of the neglect of the thermal motion of the particles. A kinetic solution is necessary which the author hopes to carry out in the near future.

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VOLKOV T. F., *Oscillations and stability of the surface of a plasma in the field of a travelling electromagnetic wave*, pp. 109–115 (completed 1958).

It is shown that the plane surface of the plasma in this case is unstable, independently of the magnitude of the external electromagnetic wave. Perturbations of the highest growth rate are those whose wave lengths are close to half the wave length of the electromagnetic wave.

BEZBATCHENKO A. L., GOLOVIN I. N., KOZLOV P. I., STRELKOV V. S. and YAVLINSKI N. A., *The high current electrode-less discharge in a toroidal chamber with a longitudinal magnetic field*, pp. 116–133 (completed 1957).

The installation with the toroidal chamber was designed and built for the study of the breakdown and the propagation of an electrode-less discharge in a magnetic field and for the study of the heating of the plasma to a temperature which, though not sufficient for the production of a thermonuclear neutrons, produces a high degree of ionization of the gas and a 'frozen-in-state' of the magnetic field.

Conclusions: Investigations have been made of the electrode-less discharge in a toroidal chamber with zero longitudinal magnetic field and with fields of up to 15,000 oersteds, electric fields from 0·15 to 6·5 V/cm, duration of the first half period of current from 350 to 2000 μsec, and with the initial pressure of the deuterium in the range 7×10^{-4} to 0·2 mm.

(1) In electrode-less discharges in a porcelain chamber the phenomena are complicated by the building up of a charge on the walls and drifts caused by crossed electric and magnetic fields. A metallic layer on the walls facilitates the breakdown, increases the conductivity of the plasma, and brings about the initial stage of the discharge into good agreement with the classical theory of conductivity and with the Townsend coefficient.

(2) The discharge without a longitudinal magnetic field starts in the centre of the discharge tube, then the column of the discharge expands and fills the whole chamber. The velocity of expansion increases with decreased pressure and with increased rate of current rise. The average conductivity of the plasma does not exceed 3×10^{13} CGS units.

(3) The discharge in the strong longitudinal magnetic field starts simultaneously across the whole cross-section of the tube and does not break away from the walls. For a field of 15,000 oersteds the discharge goes through two sharply different stages. In the field of 2500 oersteds these stages disappear. In the first stage the plasma is quiet, the radiation is uniform over the cross-section of the tube, and the conductivity grows in good agreement with the classical theory of the plasma. In the second stage there is an intense redistribution of the current density and the radiation from the discharge. The conductivity decreases. For a field of 15,000 oersteds the first stage prevails up to a current in the discharge satisfying the following condition:

$$a > \sqrt{\frac{2J_m R}{cH}}$$

where a is the radius of the plasma column, R is the radius of the toroidal chamber and J_m is the maximum current of the discharge. With a decrease of the field to 2500 oersteds the first stage continues to exist to values of a current considerably exceeding those allowed by the above criterion.

(4) The mean conductivity in the column of the discharge in a magnetic field reaches $(2 - 3) \times 10^{14}$ CGS units up to the moment of the transition from the first stage to the second.

(5) If the discharge current is continued for a long time at a value lower than that allowed by the criterion, then the conductivity reaches 6×10^{14} CGS units and then falls rapidly, probably due to outgassing of the walls.

PISTUNOVICH V. I., *The measurement of the electron temperature and ion concentration by a double floating probe in an electrode-less discharge*, pp. 134–155 (completed 1957).

The results of the work can be summarized as follows:

(1) Determination of the value of the electron temperature, ion density, and the degree of ionization of the discharge in a toroidal chamber for zero longitudinal field.
(2) It is shown that electrical fluctuations in the plasma do not strongly affect the results of the temperature measurement of the electrons.
(3) The measurement of the electron temperature and of the ion density at different distances from the centre of the chamber show that the parameters of the plasma change but little in cross-section of the toroidal chamber.
(4) The measured parameters of the plasma do not change with the orientation of the double probe with respect to the axis of the chamber.
(5) The possibility is shown of applying the double floating probe technique for temperature, measurement of electrons and ion density of the plasma for longitudinal fields up to 1000 oersteds.

STRELKOV V. S., *Investigation of the radiation of an electrode-less discharge in deuterium*, pp. 156–169 (completed 1958).

Experiments are described on the measurement of the radiation of an electrode-less pulsed discharge in the visible and X-ray part of the spectrum. The X-rays are formed by the deceleration of fast electrons in the target material or on the walls of the discharge chamber. The region of origin and the average energy of the fast electrons are determined. The time dependence of the intensity of the spectral lines of deuterium and silicon is determined in the visible part of the spectrum. This work was carried out in the toroidal chamber described in this volume, p. 116.

The change with time of the spectral intensity of deuterium lines shows that at a certain instant there is a high degree of ionization. The subsequent increase in the intensity of the spectral lines of deuterium is associated probably with the appearance of impurities from the wall of the chamber. The time of appearance of impurities is not dependent upon the magnitude of the external magnetic field and is probably determined by the time of diffusion of the impurities from the chamber walls to the discharge.

PETROV D. P., FILIPPOV N. V., FILIPPOVA T. I., and KHRABROV V. A., *A powerful gas discharge in a chamber with conducting wall*, pp. 170–181 (Work carried out in 1954 and 1957).

Results are described of the investigation of a pulsed gas discharge in a straight two-electrode chamber with conducting walls. The process of formation of the discharge for metal walls is found to differ from that for ceramic walls. The pulsed neutron emission has been investigated over a large range of gas pressures. (0·02 to 60 mm, 85 to 145 μF, 100 to 600 kA, and rise time of current 4×10^{10} to 2×10^{11} A/sec). It is shown that the source of the neutron emission is localized near the anode region of the discharge.

ANDRIANOV A. M., BAZILEVSKAYA O. A. and PROKHOROVA YU. G., *Investigation of the pulsed discharge in deuterium for values of current rise of up to* 10^{12} *A/sec and potentials up to* 120 *kV*, pp. 182–200 (completed 1958).

It is shown that for an initial potential greater than 80 kV, the duration of the neutron pulse increases significantly up to 2–3 μsec, but that the duration of the pulse of hard X-rays remains the same as for lower potentials (0·2–0·3 μsec).

BRAGINSKI S. I., GEL'FAND I. M. and FEDORENKO R. P., *The theory of the compression and oscillation of a plasma column in a strong pulsed discharge*, pp. 201–221 (completed 1957–58).

An investigation has been made of the magneto-hydrodynamic problem of the compression and oscillation of a plasma column. A system of equations with partial derivatives which describe the column, together with the equations for the discharge circuit, were integrated numerically on an electronic calculating machine. The results of the calculation are compared with experimental data.

PODGORNI I. M., CHUVATIN S. A., BIKOV G. A. and PIS'MENNI V. D., *Investigation of the process of electrodynamic acceleration of clumps of plasma* (II), pp. 222–234 (completed 1957; Part I was published previously: Artsimovich *et al. J. exp. theor. Phys.* **33,** 3, 1957).

This work contains an analysis of the experiments for determining the velocity and the size of the plasma clumps using photo-electric methods. There is also a

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comparison of the theoretically calculated curve of current with that observed experimentally.

An analysis of the current oscillograms gives for the mean velocity of the motion of the plasma clump a value of $(1{\cdot}1 - 1{\cdot}2) \times 10^7$ cm/sec.

YUSHMANOV E. E., *The radial distribution of potential in a cylindrical magnetic trap using a magnetron method of ion injection*, pp. 235–257 (completed 1957).

This article presents the results of investigations of the distribution of the potential along the radius in a cylindrical magnetic trap with magnetic mirrors and using a magnetron method of creating a plasma containing fast ions. The potential distribution was measured in its dependence on the applied potential, magnetic field, and other parameters. The measured potential distribution differed radically from the theoretical distribution corresponding to a non-compensated volume charge in the magnetron. Regular oscillations of the potential of the plasma were found under certain conditions with a frequency of 200–400 c/s. The amplitude of these oscillations was measured.

KOGAN V. I., *The broadening of spectral lines in a high temperature plasma*, pp. 258–304 (completed 1957).

A theory is developed for the broadening of spectral lines in an idealized plasma taking into account on the classical adiabatic model the simultaneous excitation of the emitting atoms by a large number of charged particles present in the discharge. General formulae are obtained for the distribution of intensity of the Stark component of the line broadened by the linear and quadratic Stark effect. These formulae have been simplified for the limiting cases of high and low densities and plasma temperature. The present shock theory was shown to be inapplicable to the broadening produced by the linear Stark effect. The broadening due to the simultaneous action of ions and electrons has been investigated. The form of the line for the case where both the Holtzmark and Doppler mechanisms of broadening operate simultaneously has been found. The results, and also the role of several other mechanisms of broadening, have been discussed as with reference to plasma in a powerful pulsed discharge in hydrogen.

TRUBNIKOV B. A., *The relation between the coefficients of absorption and emission of plasma radiation present in a magnetic field*, pp. 305–308 (completed 1957),

A short theoretical paper.

TRUBNIKOV B. A., *The behaviour of plasma in a rapidly varying magnetic field*, pp. 309–330 (completed 1957 and revised in 1958).

In this article an examination is made of the methods of thermal isolation and heating of a plasma with the help of external rapidly varying magnetic fields. It is shown that by a suitable choice of the geometry of the field, the plasma can be in equilibrium and therefore stable. The problem of heating the plasma is partially examined.

MOROZOV A. I., *Cherenkov generation of magneto-acoustical waves*, pp. 331–352 (completed 1958).

An investigation has been made of the generation of magneto-acoustical waves in both finite and infinite plasma. The source of the wave is considered to be circuits carrying currents and moving at high velocities. In the work of B. A. TRUBNIKOV (p. 309), using the hydrodynamic approximation, the problem of sound generation has been examined in compressible semi-bounded magnetized medium by excitation of pressure travelling along the surface.

In high temperature plasma without a magnetic field the results obtained by TRUBNIKOV are not applicable in as much as sound cannot propagate in such a plasma because of the long mean free path of the ions.

If, however, inside the plasma there is a magnetic field, then, as is well known, the dispersion equation contains a branch corresponding to magneto-acoustical waves. (cf. BRAGINSKI, USSR, Acad. of Sci. **115,** No. 3, 1957). These waves are obtained both for large and for small mean free path of the particles, if the frequency of the wave is much smaller than the Larmor frequency of the ion. (cf. GINZBURG, *J. exp. theor. Phys.* **21,** 788, 1951). If the frequency of collisions of ions with neutral particles is much larger or smaller than the frequency of the wave, then the magneto-acoustical waves decay weakly. Under these conditions, the magneto-acoustical waves, can be described also by a system of magnetohydrodynamic equations for an ideal liquid and this fact is used in this article.

The question of generation of magneto-acoustical waves has a definite interest in connexion with the problem of heating the plasma and, likewise, in connexion with working out the details of corrugated magnetic systems inside which the plasma moves.

KADOMTSEV B. B., *Magnetic traps for plasma*, pp. 353–363 (completed 1957).

In this work a hydrodynamic investigation is carried out of magnetic traps. It is shown that

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systems with guard conductors are also magnetic traps, the losses from which are due basically to the suspensions. The presence of stability also has been established in traps with corrugated fields.

KADOMTSEV B. B., *The instability of plasma in a magnetic field in the presence of ionic beams*, pp. 364–369 (completed 1958).

This is a theoretical paper. In some cases in the plasma, there may be directed beams of ions. Thus, in the plasma of an ion magnetron created by accelerating ions in a radial electric field, all ions undergo approximately the same periodic motion: moving in a radial direction they change their direction under the action of the magnetic field until finally they return to the central region of the plasma. Such a motion leads to the formation of two beams of ions meeting each other in radial directions.

An analogous situation may arise in the adiabatic trap with magnetic mirrors for the production of plasma by means of accumulation of ions from an ion source. Such a plasma occupies a ring region with a thickness equal to the diameter of the ion ring. But the ions again form two counter-currents in radial direction.

As is well known, electron beams in a plasma may be unstable. An analogous situation can take place in this case.

In this article is investigated the instability of plasma with respect to short wave perturbations, the wave length of which is much smaller than the mean Larmor radius of the ions.

It is shown that the beam of ions in the plasma will be unstable if the spread of ion velocity is very small.

KADOMTSEV B. B., *The dynamics of plasma in a strong magnetic field*, pp. 370–379 (completed 1958).

It is shown that for motion of the plasma, slow compared with the ion cyclotron frequency, the motion can be described by the equations of hydrodynamics for the motion across the magnetic field, and kinetic equations for the transverse motion. These equations can be simplified partially for the case of a plasma the pressure of which is much smaller than the magnetic pressure.

Examination is made of the question of stability of the plasma of low pressure in a corrugated field.

KADOMTSEV B. B., *The convective instability of a plasma*, pp. 380–383 (completed 1958).

This article investigates the instability of equilibrium magnetohydrodynamic configurations with closed lines of force for a definite form of perturbation.

SAGDEEV R. S., *Non-linear motions of a dilute plasma in a magnetic field*, pp. 384–390 (completed 1958).

A solution has been found for established magneto-acoustical waves of large amplitude propagated across the magnetic field.

MOROZOV A. L. and SOLOV'EV L. S., *The damping of oscillations of a plasma column*, pp. 391–414 (completed 1958).

The problem of the damping of the oscillations of a plasma cylinder using various shields placed on the surface of the plasma has been examined.

The different cases of the oscillations of the plasma investigated in this work show that these oscillations can be damped effectively by using the appropriate casing. If this is of a non-conducting material, the damping takes place via the Cherenkov effect. For large damping under the conditions of stability, it is necessary that the ratio of the distance of the plasma to the shielding, to the dielectric constant be several times (2–10 times), smaller than the period of the wave.

If the casing is conducting, for effective damping it is necessary that the skin layer corresponding to the frequency of oscillation of the wave be several times smaller than the above ratio.

It is noted that it is not necessary to damp the oscillation with the help of a conducting layer, that is, using the Joule heating of this layer. One can also damp the waves by means of the Cherenkov effect, in which the oscillation energy of the plasma is removed in the form of high frequency electromagnetic waves.

The remarks made above would appear to deal only with the stabilization of high frequency oscillations of the plasma. The slow 'creeping away' of the plasma column can be stabilized by internal fields of the type proposed by OSOVETS (Vol. 2, p. 238) or by thick copper screening. The layer damping the oscillations can be put on the internal surface of such a shield.

It is to be noted in conclusion that the damping layer may effectively iron out the rotational electrical fields inside the toroidal chamber with a copper casing containing gaps, and in this way serve as a liner, if its vacuum properties turn out to be sufficiently good.

SHAFRANOV V. D., *Deduction of the tensor of dielectric constant of a plasma*, pp. 416–421 (completed 1958).

In the present note there has been obtained by means of the integration of the Boltzmann equation along the particle trajectories, a general expression for the tensor of dielectric permeability of a plasma in the absence of collisions.

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SAGDEEV R. Z., *Absorption of electromagnetic waves propagating along a constant magnetic field in a plasma*, pp. 422–425 (completed 1958).

An investigation is made of the problem of the absorption of electromagnetic waves in a plasma due to the thermal motion of electrons.

It is of interest to analyse the mechanisms of the absorption of the energy of the plasma under conditions when two particle collisions can be neglected. This is particularly essential for high temperatures when the Joule absorption is sharply decreased and it is necessary under these conditions to look for more effective mechanisms, not associated with colliding particles.

Solutions are given of the problem of the propagation of the electromagnetic wave along a constant magnetic field without linearizing the problem, but only neglecting the action of the magnetic field of the wave on electrons with fairly general initial conditions. This analysis can be considered as a particular case of the results of the linearized theory.

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SHAFRANOV B. D., *The index of refraction of a plasma in a magnetic field near the ion cyclotron resonance frequency*, pp. 426–429 (completed 1958).

In connection with the problem of dissipation of energy in a plasma in the absence of collisions (p. 422), it is of interest to determine the conditions under which there is an absorption of energy of the electromagnetic field by the particles of the plasma, (the range of frequencies, direction of the propagation of the field, etc.). For determining the region of absorption of the electromagnetic field by the ions, the square of the index of refraction of the plasma is calculated, taking into account the motion of the ions along the lines of force of the magnetic field.

SAGDEEV R. Z. and SHAFRANOV V. D., *The oscillations of a plasma column taking into account the thermal motion of the ions*, pp. 430–435 (completed 1958).

As is well known, on increasing the temperature of a plasma the effectiveness of the usual Joule mechanism for heating falls. It is of interest to investigate other methods of heating of the plasma. One of these methods is the 'betatron' method of heating, based on the periodic change of the magnetic field, with a period comparable with the time between ion collisions.

In the high temperature plasma in which collisions are rare it is interesting to investigate the mechanisms of absorption of energy which are not associated with collisions. The physical basis of such non-collision (cyclotron and Cherenkov) absorption are discussed in **4**, pp. 422 and 426 and *J. exp. theor. Phys.* **34**, 1475 (1958). For a calculation of the absorption of energy in a plasma column it is necessary to solve the problem of the oscillations of the plasma column. The oscillations of the plasma cylinder at frequencies considerably smaller than the cyclotron frequencies of the ions were investigated by TRUBNIKOV (**1**, p. 289) and SHAFRANOV (**2**, p. 130). The cyclotron resonance of ions was considered by STIX (*Phys. Rev.* **106**, 1146, 1957), while the study of the high frequency oscillations was carried out by KORPER (*Z. Naturf.* **12a**, 815 1957) for the case of oscillations along the axis. All these articles were based on a hydrodynamic description of the plasma. RUDAKOV and SAGDEEV solved the problem of the oscillations of a plasma column starting out from kinetic equations in the drift approximation, which takes into account thermal motion of the ions along the lines of force of the field for frequencies less than the frequency of the internal magnetic field. Such an approach gives a possibility of considering Cherenkov absorption.

In the present work a solution is given of the problem of the oscillations of a cylinder of plasma taking into account the thermal motion of the ions leading both to Cherenkov and cyclotron absorption.

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In view of the fact that not all the articles appearing in the *Soviet Journal of Atomic Energy* are of interest to our readers, the Board of Editors have decided to publish only such Russian papers which they consider to be within the scope of the *Journal of Nuclear Energy*. However, arrangements have been made, with the kind assistance of the Pergamon Institute (a non-profit-making foundation) and the Academy of Sciences of the U.S.S.R., for the *Journal of Nuclear Energy* to carry translated contents lists and abstracts of the complete forthcoming issues of the *Soviet Journal of Atomic Energy*. A translation of any article which has not been chosen as suitable for inclusion in the *Journal of Nuclear Energy* may be obtained on request from the Pergamon Institute, under their co-operative cost-sharing plan.

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ON THE STUDY AND DESIGN OF CYCLIC ELECTRONIC ACCELERATORS AT THE TOMSK POLYTECHNIC INSTITUTE*

A. A. VOROBIEV and V. A. MOSKALEV

Abstract—A short account is given of the results of studies on cyclic electronic accelerators performed since 1946 at the Tomsk Polytechnic Institute. The basic design data and special constructional features of several betatrons are described. Various methods are discussed for extracting the electrons from the betatron accelerating chamber. A betatron producing 25 MeV electrons is described in more detail. The use of annular accelerating electrodes connected to an external two wire resonant transmission line in a 30 MeV synchrotron is also reported.

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INTRODUCTION

IN 1946 studies and design work on betatrons were begun at the Tomsk Polytechnic Institute (TPI). In 1948 both a 5 MeV betatron operating with a coil current of 500 kc/s frequency and a betatron with up to 7 MeV γ-ray energy using mains frequency current were constructed. In the period from 1949 to 1955 a series of betatrons was completed having energies up to 15 MeV. At present these betatrons are operating successfully in scientific research institutions throughout the country. While this series was being constructed a betatron of up to 25 MeV γ-ray energy was being designed which has an increased intensity of radiation and in which the loading of each element of the iron circuit is balanced. Several such betatrons were completed in 1955 and 1956.

In this article we describe briefly the results of some studies on electron accelerators performed at TPI and also some design details of one of the betatrons, that of 25 MeV γ-ray energy.

RESULTS OF SCIENTIFIC AND TECHNICAL STUDIES ON ACCELERATORS

The construction of the accelerators was made possible by the theoretical and engineering work of the whole staff of the Institute.

An original theory of how electrons are caught up in the accelerating process was proposed by MELIKHOV[1] and was confirmed by experiment. The essence of this theory is as follows. As is well known, the electron gun feeds electrons into the betatron in short bursts. As the injection voltage rises (at the leading edge of the pulse) the radius of the instantaneous orbit into which the electrons are fed is increased and the amplitude of their oscillations decreases. During injection therefore, the space charge density is not uniform across the beam and increases with time towards the periphery. This results in an inward displacement of the instantaneous orbits, so that they avoid the electron gun. Calculations show that the amount of orbit displacement can be very great especially when the current from the electron gun is large. This results in electrons hitting the internal wall of the chamber. On the other hand, for small currents the orbits may be displaced insufficiently to avoid the injector.

If the injection voltage drops, the instantaneous orbits are displaced outwards and the electrons are bound to hit the gun. MELIKHOV obtained oscillograms for the variation of the current in the accelerating volume at the moment of injection. He showed that if the electron gun is situated outside the equilibrium orbit the electrons are caught up in the accelerating process essentially at the leading edge of the burst.

CHUCHALIN[2] used an injection voltage with two peaks and so was able to double the radiation yield of the betatron.

It was established that:

(1) Electrons may be caught up in the acceleration process both when the injection voltage rises and when it falls.

(2) Each time electrons are caught up, during a few revolutions, this is followed by the loss of a large fraction of the accelerated charge. (One can assume that this is essential to enable the remaining electrons to get past the injector).

(3) If the voltage pulse on the injector is of an 'idealized' form, which matches the curve of voltage necessary for the process, an increase of the accelerated charge and hence of the yield of radiation can be achieved by applying to the injector a voltage with several peaks, such that the curve of the necessary voltage is repeatedly intersected by the curve of voltage applied. These intersections must, however,

* Translated from *Atomnaya Energiya* **4**, 229 (1958).

take place with a definite spacing in time which is sufficient for the electrons caught up due to the preceding voltage peak to have formed a stable toroidal bunch which is not disturbed by the charges caught up as a result of subsequent peaks in the injection voltage.

FILIPPOV[3] proposed a method for choosing uniquely the radius of the equilibrium orbit in a betatron given a maximal energy for the accelerated electrons. He obtained a linear equation expressing the dependence of the equilibrium orbit radius r_0 on the parameters of the magnetic field and the size of the pole gap:

$$r_0 = BE + b_0,$$

where E is the calculated maximum electron energy, b_0 half the radial dimension of the doughnut cross-section and B the 'betatron constant' which depends on the properties of the magnet system but remains constant for betatrons with different electron energies.

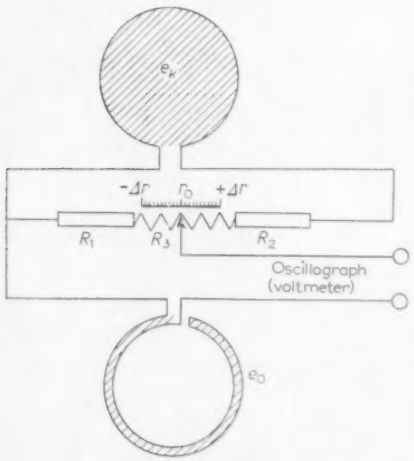

FIG. 1.—Diagram of arrangement for determining the radius of the equilibrium orbit.

The equation expresses the radial focussing properties of the magnetic field and determines the tolerances in the construction and assembly of the elements of the magnet system. It also enables one to calculate the changes in dimensions and parameters required to adjust the electron orbit.

The method used for continuous control of the position of the equilibrium orbit was developed by LESHCHENKO[4] both for annular and for race-track type accelerators. A system of coils connected according to the scheme given in Fig. 1 is placed both above and below the doughnut. The e.m.f. e_k produced in one of the coils is proportional to the mean value of the magnetic induction within the orbit while the e.m.f. e_0 in the other coil is proportional to the instantaneous value of the induction at the orbit. When the magnet is fed continuously a sensitive valve voltmeter is used as a null indicator, when it is fed in bursts, an oscillograph is used. This arrangement allows changes in the equilibrium orbit radius to be observed with an accuracy of 1 mm.

To measure phase inhomogeneities of the magnetic field ANAN'EV[5] devised a micro-phasometer, which achieves improved sensitivity by using pulse shaping and 'amplification' of small phase differences.

A pulse generated in a probe in the field is shaped and then works a trigger with variable effective resistance (a diode and high ohmic resistance in parallel). A pulse from a second probe works another trigger which generates a rectangular pulse of long duration (5×10^{-3} sec). This pulse is fed to the anode of the diode and so changes the constants of the circuit determining the length of the pulse from the first trigger. The time during which the rectangular pulse affects the anode of the first trigger circuit therefore depends on the moment when the diode is blocked, i.e. on the phase difference between the magnetic fields at the positions of the two probes. This micro-phasometer amplifies phase differences up to 26 times. The readings are displayed on the scale of a micro-ammeter which extends to 10 mA. A deflexion over the whole scale corresponds to a phase difference of 10 μsec between the two points in the magnetic field.

ANAN'EV[5] also calculated theoretically and verified experimentally the influence of inaccuracies in the construction of various parts of the electromagnet on the operation of a betatron. He obtained equations relating the relative changes of the radius r_0 of the equilibrium orbit and of the index of the fall off of the field to the inaccuracies in the dimensions of the electromagnet. He showed that r_0 is most strongly affected by inaccuracies in the shaping of the pole pieces, which may give rise to deviations of 10–40 per cent from the calculated value of r_0. To simplify the process of alignment he proposed a method of regulating the orbit radius by displacing the central magnet core. These investigations made it possible to develop several original features of betatron construction.

NOSKOV[7] and DIMOV[6] designed and constructed two variants of a pulsed betatron of up to 10 MeV energy with an iron-free magnet system. This betatron consists of two coaxial coils and a central core which is bisected in the median plane. The coils are fed by a pulsed current giving between 1–2 and 20 pulses per second. For the same energy, the weight of the

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betatron is about 1/10 of that of a betatron with iron magnets.

MOSKALEV[11] designed a 15 MeV betatron for the treatment of cancerous tumours. It has a lead collimator which produces a narrow beam of γ-rays, and an exchangeable central piece by means of which the size of the irradiated area can be adjusted. The variation of the dose across the beam is removed by inserting a specially designed conical copper filter into the beam path. The properties of the beam and the iso-dose curves in organic tissue were studied by means of a water dummy specimen. The deep dose obtainable with the betatron is considerably greater than that provided by X-ray apparatus. The surface dose is several times smaller than the maximum dose observed at a depth of 20 mm. This apparatus was used by members of our Institute together with workers at the Tomsk Medical Institute in animal experiments which gave interesting results.

In 1955 a synchrotron with an electron energy of up to 30 MeV was started up at the TPI, in which a new resonance system was used. In most of the electron synchrotrons in operation, the accelerating devices are coaxial resonators, which have a number of serious disadvantages. (Relatively low quality factor, owing to the limited space available for the resonator; difficulty in modification for frequency alteration, etc.). With the accelerating device proposed by VOROBIEV it is considerably easier to change the frequency; it consists of a two wire transmission line, capacity loaded, between two ring electrodes surrounding the chamber over the length of the accelerator gap, enclosed in a conducting sheath. As such an arrangement takes up very little space within the pole gap, all geometrical dimensions may be given their optimum values. This allows one to obtain greater values of the quality factor and of the dynamic resistance than one does with synchrotrons using a coaxial resonator. The motion of electrons in the alternating electric field of the electrode rings was analysed and it was shown that the acceleration is a difference effect. Part of the energy gained by an electron passing through the accelerating gap is lost along the rest of the orbit. It was established experimentally, however, that the energy lost in one revolution is considerably less than the energy acquired in passing through the accelerating gap.

An alteration of frequency can be effected by moving a short-circuiting bridge on the transmission line. This system is symmetric and is convenient if the h.f. generator has a balanced output, so that symmetrizing devices are unnecessary. To match the input resistance to the feeder the point at which the feeder is joined to the transmission line is varied. Finally, this arrangement is much simpler to construct than any type of coaxial resonator. The system has the disadvantage of radiating electromagnetic energy, but this radiation can be considerably reduced by screening.

The extraction of the electron beam by means of a deflecting condenser was achieved by KONONOV.[10] A coil wound symmetrically so as to increase the magnetic field near the centre of a pole piece and thus to destroy the 2–1 betatron condition, causes the size of the electron orbit to increase. The shift of the orbit gradually increases and by the time the electron enters the deflecting condenser the shift amounts to 2 mm per revolution. The deflecting condenser consists of two plates about 5 cm in length bent to match the radius of curvature of the orbit at the position of the condenser. The internal (screening) plate, is earthed and the external (deflecting) plate receives a voltage pulse of 20 to 40 kV, according to the energy of the accelerated electrons. The voltage pulse is taken from a pulse transformer. The length of the effective part of the pulse is 300 μsec. The deflexion condenser receives its pulse slightly sooner than the coil that expands the orbit, so that it carries the full working voltage when the electrons enter the condenser.

The electrons leave the doughnut through a 30μ aluminium foil. The extracted beam is well focussed and at 50 mm distance from the window its cross-section is 5×8 mm^2. At 50 cm the cross-section of the beam increases to 12 cm in width and 6 cm in height. These figures refer to 5 MeV electrons. At 10 MeV and above the beam focusing is better.

The beam current, measured with a Faraday cylinder and by a chemical method, is about 4×10^{-9} A. The beam intensity measured with a thimble ion chamber near the exit window is 300 r/min. Electromagnetic extraction of the beam was achieved by SOKOLOV using two different methods.

The first method used a magnetic shunt placed along the tangent of the electron trajectory which at the instant of extraction is of the form of an outward spiral. The effectiveness of the magnetic shunt is not high, the method permitting the extraction of about 2 per cent of all accelerated electrons, giving a current of 3×10^{-11} A.

Such a small efficiency is due to three circumstances.

(1) The presence of the shunt between the pole pieces excites radial oscillations of the electrons even before they enter the shunt channel, owing to the weakness of the main magnetic field near to the shunt. This results in the loss of a certain fraction of the accelerated electrons.

(2) The vertical spread of the beam at the instant of extraction is considerably greater than the vertical dimension of the shunt channel so that in fact only a small fraction of all electrons enter. It does not help to increase the height of the shunt channel because this drastically lowers the efficiency of the shunting.

(3) The presence of the steel shunt in the doughnut has a harmful effect at the moment of injection on the process of accelerating electrons so that it lowers the yield of the betatron.

The second method for electromagnetic extraction of electrons uses sector coils placed outside the doughnut over and under the equilibrium orbit and fed by a pulsed current. This method has not hitherto been described in the literature. The current pulse in these coils weakens the magnetic field by 10 per cent at azimuth angles up to 90°. The extracted electron beam has a divergence angle of 70°. The electrons in the beam are not uniformly distributed. There can be one, two or more bunches of electrons spreading within the limit of angles indicated, depending on fringe field effects.

According to the theoretical ideas we have worked out the extraction of the electron beam from the doughnut may be considered as a limiting case of a displacement of the accelerated electron from the equilibrium orbit. By simultaneously using coils wound so as to produce a symmetric expansion of the orbit and other coils acting unsymmetrically on separate segments at particular azimuth angles, up to 75 per cent of the accelerated electrons could be extracted.

Recently two 15 MeV betatrons with external electron beams have been installed in scientific research institutions in Moscow.

The construction of a 30 MeV betatron of small dimensions for use in medicine and also for the detection of defects in large metal articles in factory conditions was proposed by ANAN'EV.[5]

The construction of the pole pieces of the betatron enables it to operate with a magnetic induction in the central core which is greater than the saturation induction of the magnetic material. Such pole-piece construction makes it possible to decrease the weight of the electromagnet by a factor of 2–2·5. To reduce losses in the steel, a pulsed feed is employed and the central core is demagnetized by a pulsed current. The demagnetizing current pulse is 5 to 10 times longer than the pulse of the main current. To reduce the dimensions of the arrangement, the demagnetizing coil is fitted into the body of the pole. As a result, the radius of the equilibrium orbit and the dimensions of the doughnut are not reduced.

The best utilization of the magnetic properties of the material and the reduction of the dimensions of the coils, when pulsed feeding is used, result in a saving of 30–40 per cent in the weight of the electromagnet. The electromagnet is of symmetric shape forming a closed cylinder. The dimensions of the system are as follows: height of the electromagnet 42 cm, diameter, 59·5 cm, power consumption 6·5 kV, weight about 1000 kg. A machine of the same energy developed in Switzerland weighs 7 tons.

MOSKALEV[11] suggested and constructed a stereo-betatron with two accelerating chambers and without any branching of magnetic flux; it is intended for use in medicine and in the detection of metal defects.

This betatron is in essence a combination of two betatrons constructed as one and produces four beams which intersect at one point. In its medical use this allows the irradiation of a deep tumour from four directions, so that the radiation dose applied to the tumour can be increased fourfold without increasing the surface dose.

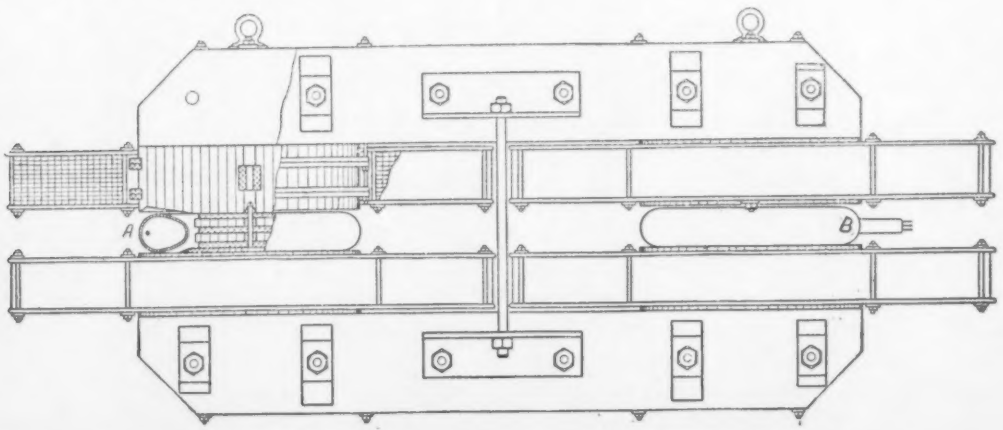

FIG. 2.—The electromagnet of the stereo-betatron. *A*, *B* possible positions for two targets.

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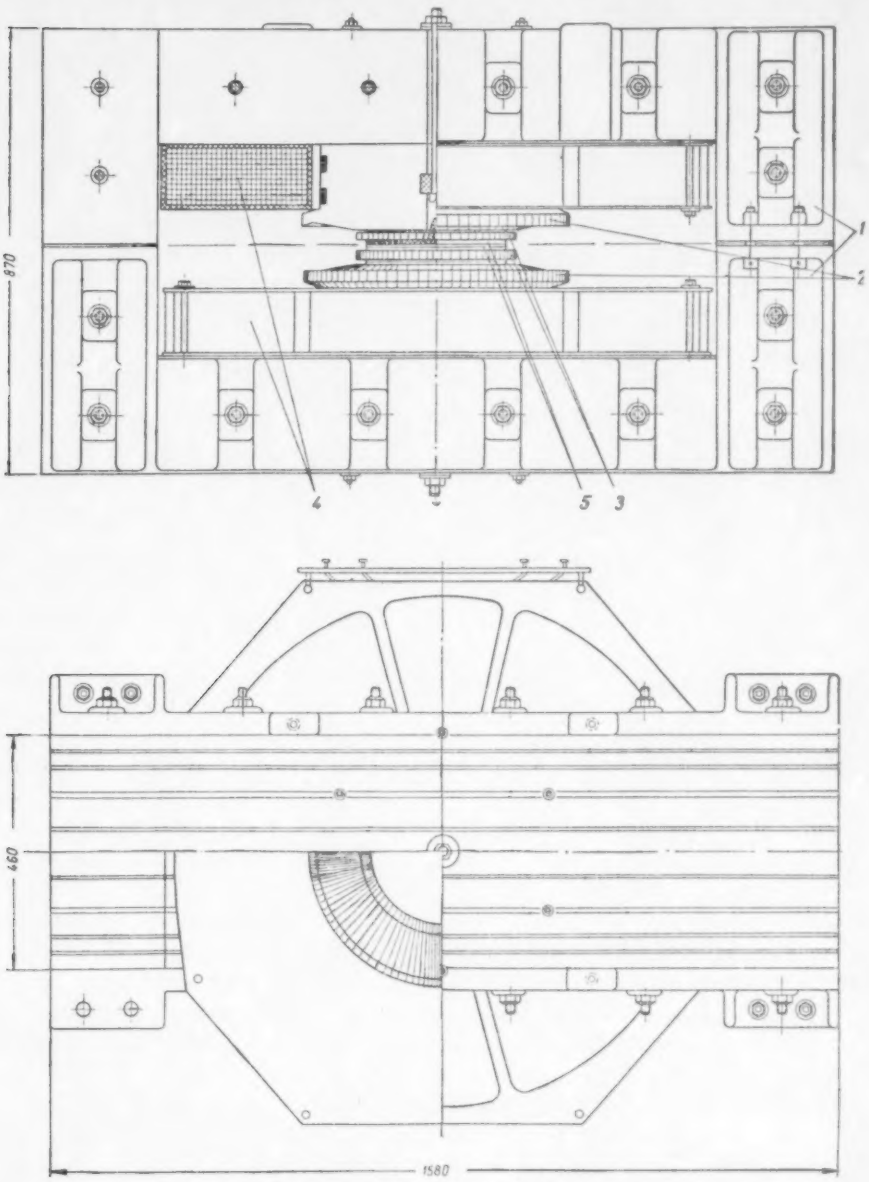

FIG. 3.—Electromagnet of the 25 MeV betatron.
1. Magnet yoke 2. Poles 3. Inserted disks. 4. Coil 5. Textolite layer.

It is also possible to get the beams to intersect in pairs, one point of intersection on each side of the apparatus. In studies of metal defects in industrial production stereo-photographs of two articles can be made simultaneously. The long base line of each pair of beams makes it possible to determine very accurately the depth of a metal defect.

GORBUNOV[12] used a betatron for laboratory study of metal defects in articles of great thickness. Dosimetric apparatus specially developed enables one to determine automatically the dose necessary for a given article. Tables and graphs have been produced giving the exposures required to give various degrees of photographic plate blackening for various thicknesses of specimen.

In addition to those described, other accelerators are being developed at the TPI, including accelerators of greater yield, small size betatrons for studying metal defects in factory products under industrial conditions and a pulsed betatron of very high radiation intensity.

The betatrons developed at the TPI are comparatively cheap, reliable and stable in operation. Their maintenance is simple. They can be employed

successfully in the country's industry and in medical work, where, as is well known, betatrons have valuable scientific and technical uses.

DETAILS OF THE BETATRON OF 25 MeV ELECTRON ENERGY

The electromagnet of this betatron (Fig. 3) has an E-shaped yoke made of electro-technical sheet steel E42 (plate thickness 0·35 mm). The sheets are insulated with bakelite varnish. The profiled pole pieces consist of stamped sheets of the same material. The plates are positioned radially. The whole pole consists of 224 sectors with 19 plates in each sector (Fig. 4). Two disks of radial steel plates are inserted into the central region of the pole gap.

The electromagnet operates at mains frequency so that complicated and costly high frequency generators are unnecessary. As a result the circuit scheme is simple (Fig. 5) and reliable in operation. The mains frequency is fed into the primary winding of the magnet. The effective magnetic flux is produced by

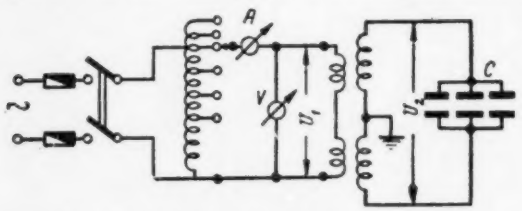

FIG. 5.—Feed circuit for the betatron magnet.

the secondary (magnetizing) winding the circuit of which includes a condenser battery, C, of 70 μF capacity (power of about 100 kVA) which serves to compensate the reactive power consumed by the machine. When the voltage across the terminals of the secondary winding, U_2, is 6 kV, the current is about 130A. The induction is then 13,100 gauss in the steel of the central disks and 9000 gauss in the yoke. The profiled region of the air gap extends to 11·5 cm in the radial direction. The height of the air gap at the equilibrium radius r_0 ($r_0 = 21$ cm) is 7 cm. The steel of the magnet weighs 3200 kg. The index of the fall-off of the magnetic field in the air gap is 0·75.

In Fig. 6 graphs are given of the azimuthal variation δH of the magnetic field.

Along the equilibrium orbit the azimuthal variation of the field has a clearly discernible first harmonic which has a value of up to 8–10 gauss unless a correction is made. The azimuthal variations are decreased by means of special compensating coils placed in vertical stacks so as to modify the magnetic field of the yoke.

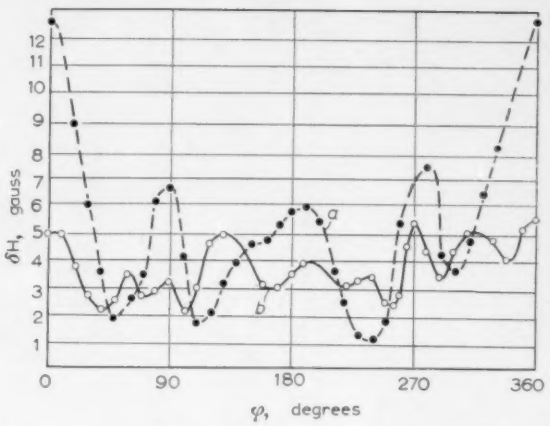

FIG. 6.—Graph of the azimuthal variation of the magnetic field of a betatron.
a—before correction.
b—after correction.

Recently a method has been used in which the magnet yoke assembly is loaded uniformly with magnetic flux; this is achieved by splitting the yoke suitably into separate regions. In this case, the azimuthal inhomogeneity of the field decreases 5 to 7 times and no additional correction is needed. This method has been worked out by FILIPPOV of this Institute and has not been published before.

The doughnut is of porcelain or glass (Fig. 7) and has branch pipes for connexion to the vacuum pumps and for the electron injector. The cross-section of the doughnut is ellipsoidal with axes of 11 and 5·6 cm.

Fig. 8 shows schematically the three electrode electron injectors used in the betatron. Fig. 9 shows the circuit for pulsed operation of the injector. The

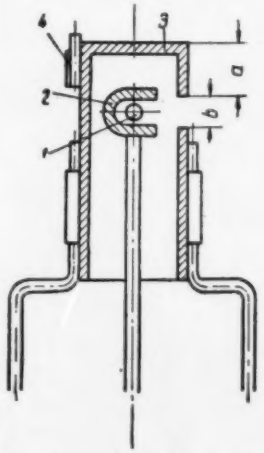

FIG. 8.—Schematic diagram of the three electrode betatron injectors.
1. Cathode. 2. Focussing electrode. 3. Anode.
4. Target. $a = 1$ mm $b = 1·1$ mm.

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FIG. 4.—Betatron magnet pole-piece.

FIG. 7.—Glass doughnut with attached injector and deflector. The aluminium extraction window is at the front of the picture.

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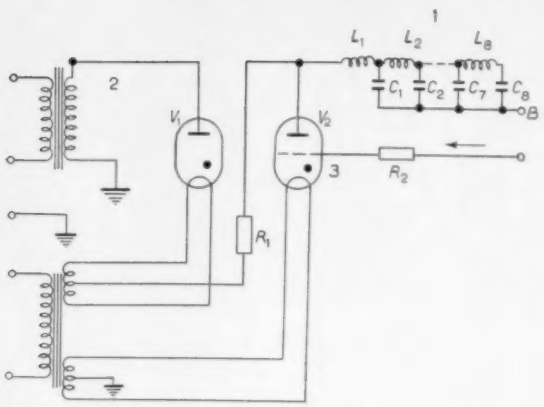

FIG. 9.—Circuit scheme for pulsed operation of the betatron injector.

pulse shaping line 1 is charged from the rectifier 2. At the required moment a pulse from a permalloy probe in the magnetic field of the betatron ignites the thyratron 3 and the condensers of the pulse shaping line are discharged through the primary winding of the pulse transformer. The voltage pulse, transformed to 30–35 kV is applied to the cathode of the injector. The active period of the injector is determined by the active period of the cathode.

At the end of the accelerating cycle a current pulse

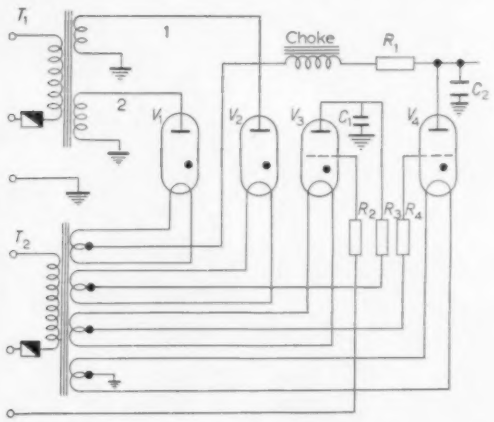

FIG. 10.—Circuit scheme for the coils displacing the electrons from the equilibrium orbit
V_1: *VG* 213; V_2: *VG* 129; V_3: TG1 400/16.

in a special coil, either on the central core or the poles above and below the doughnut, destroys the betatron condition so that the electrons spiral outwards and are thrown on to the target, a tungsten plate attached to the rear side of the injector anode. In the target, the electrons are decelerated and emit extremely hard radiation.

The circuit which feeds the current pulse through the deflector coils is given in Fig. 10. The condenser C_1 is charged from a discharge tube rectifier 1 (V_2) up to 200–250 V and the condenser C_2 up to 4 keV from rectifier 2 (V_1). A gating pulse reaches the grid of the thyratron V_3 from the permalloy probe via a delay circuit, the condenser C_1 discharges through the thyratron V_3 and the resistance R_4 after the gating pulse reaches the grid of thyratron V_4. Condenser C_2 discharges via thyratron L_4 through the deflecting coil. All the electronic circuits are assembled in units and mounted on a small control desk.

The radiation yield of this betatron reaches 60 r/min at 1 m from the target. The 25 MeV betatron is installed in the Tomsk Medical Institute and is used for clinical and research purposes.

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NOTE

A negative-ion injector for a charge-exchange electrostatic generator*

A new type of negative ion injector has been developed by YA. M. FOGEL, A. G. KOVAL and A. D. TIMOFEEV at the Physico-technical Institute of the Ukrainian Academy of Sciences. A powerful source of positive ions, producing a current of up to 7 mA, is used in conjunction with a charge-exchange target consisting of a supersonic jet of mercury vapour in vacuum. At the output a mass spectrometer selects mono-energetic negative ion beams, each of fixed mass. The instrument is shown in an accompanying figure. Negative-ion beams of the following intensities have been obtained: hydrogen, 70 μA; helium, 0·17 μA; oxygen, 40 μA.

V. P.

* Translated from *Atomnaya Energiya* **5**, 590 (1958).

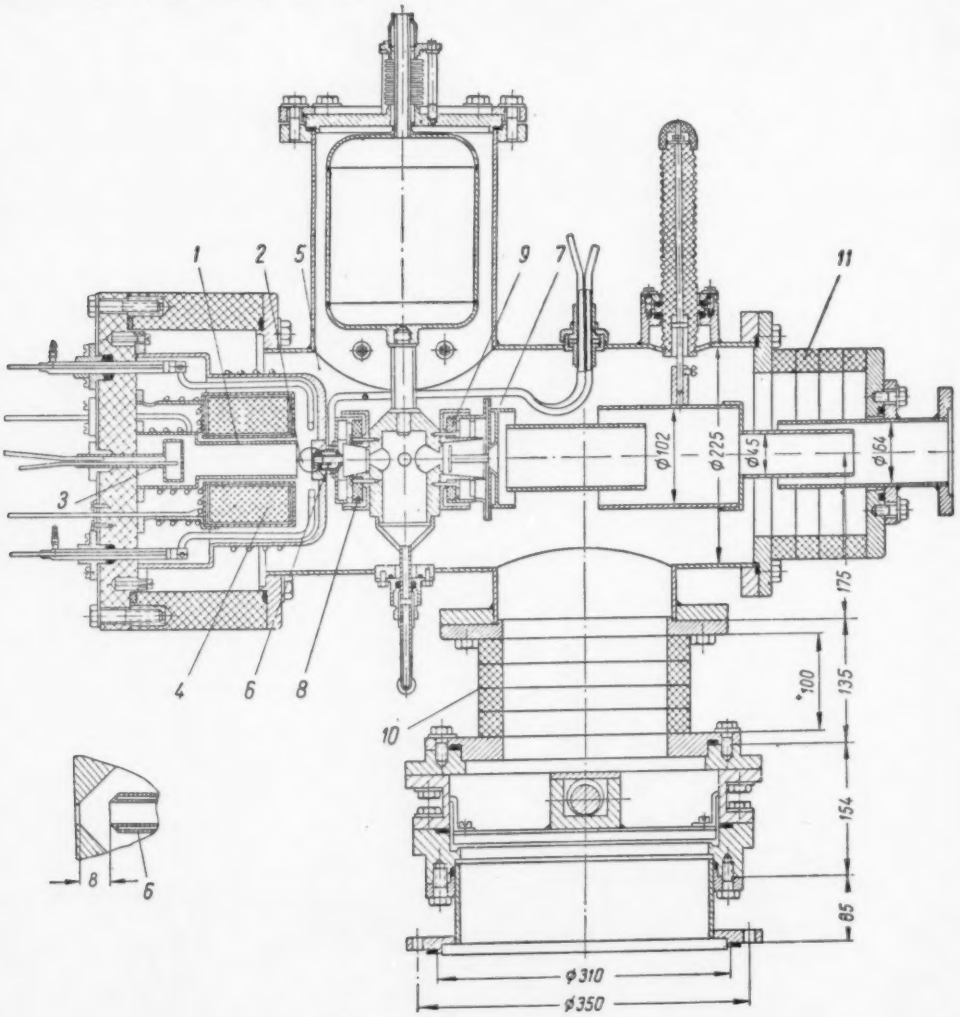

FIG. 1.—The negative-ion injector.
1—anode of the ion source; 2—spiral cathode; 3—electron repeller; 4—source magnet coil; 5—source housing; 6—ion extraction electrode; 7—beam outlet diaphragm; 8—insulator for the input channel; 9—insulator for the outlet diaphragm; 10—insulating connexion to the vacuum system; 11—insulating connexion to the mass-analyser.

J. Nucl. Energy, Part C: Plasma Physics, 1959, Vol. 1, pp. 97 to 104. Pergamon Press Ltd. Printed in Northern Ireland

CURRENT PAPERS IN PLASMA PHYSICS, ACCELERATORS AND THERMONUCLEAR RESEARCH

Each issue of this journal will include a list of the titles of papers and reports on plasma physics, accelerators, and thermonuclear research, appearing in some 50 to 100 journals published throughout the world. The Editors intend this list to be as complete as possible, and would be glad to hear of any important omissions so that these may be included in the next issue. The present list includes publications appearing since 1st January 1959.

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(i) Current Papers in Plasma Physics and Thermonuclear Research

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RUNAWAY AND SUPRATHERMAL PARTICLES

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Abstract—Under certain conditions an electron or ion is accelerated continuously when subject to a force greater than the dynamic friction; as a result it is decoupled from its thermal state of motion and is said to be 'runaway'. Runaway particles lead to, and form part of, a class of particles which have been referred to elsewhere as 'suprathermal'. The study of the way in which runaway particles are produced and accelerated is not only of importance in laboratory gas discharges, but also in a wide variety of applications in astrophysics. The subject of runaway electrons and ions is treated, first in a uniform ionized gas in which the particles have Maxwellian velocity distributions and, secondly, in a current conducting plasma. The conditions for decoupling and accelerating particles from their thermal state are discussed and, in particular, it is shown that in the case of a weak electric field ions are unlikely to achieve speeds very much in excess of the drift speed of the electrons, except in restricted regions where runaway electrons are continually forming and escaping. When the ratio of the thermal and magnetic energy densities, given approximately by β, is small, the runaway electron current grows rapidly at the expense of the conducting current. A number of ways are considered in which the runaway current may grow and become predominant. Finally, the generation of suprathermal ions by Fermi interactions is considered, from which it is evident that when β is the order of unity the ion gas is heated and when β is small relatively few ions are accelerated to high energies.

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1. INTRODUCTION

A charged particle moving in a plasma is subject to a retarding force due to collisions and the excitation of collective oscillations; this retarding force is referred to as the dynamic friction (CHANDRASEKHAR, 1943). When the particle, initially in a thermal state of motion, gains more momentum from an applied force than is lost through the dynamic friction, it is accelerated; if now the dynamic friction decreases with increase in velocity, the particle is continuously accelerated and is then said to be 'runaway' (WILSON, 1925; EDDINGTON, 1926). Once the particle has gained by this process an energy considerably in excess of the mean thermal energy, it becomes decoupled from its original thermal state and can be said to be 'suprathermal' (PARKER, 1958a). It may happen that after the particle has run away into a suprathermal state, the dynamic friction increases as the particle energy increases; in this case it ceases to be runaway, but remains suprathermal so long as the applied force is maintained. In general, runaways form a special class which either lead to or form part of the suprathermals.

The study of the production of suprathermal particles is of particular interest astrophysically in a wide variety of applications (PARKER, 1957; 1958a,b,c; PARKER and TIDMAN, 1958a,b,c). The effect of such particles in laboratory discharges in the past has been less pronounced owing primarily to the small probability of their containment; improved methods of containment in the future, however, may elevate the subject of suprathermal particles to one of prime importance in this field.

The subject of runaway electrons in applied electric fields has been dealt with by a number of authors (GIOVANELLI, 1947, 1948, 1949; HOYLE, 1949; GIBSON, 1956, DREICER, 1956; 1958; LINHART, unpublished; HARRISON, 1958) who have assumed Maxwellian velocity distributions for the ions and electrons in the thermal state. No matter how small the applied electric field, it appears that there are always electrons in the high energy tail which qualify for the runaway state, and which therefore continuously contribute to what is virtually a 'supercurrent' (LONDON, 1950). It is shown in this paper that the runaway electron current will remain small in comparison with the conduction current in the time taken for the magnetic field to decay provided the energy density of the magnetic field is not too large as compared with the thermal energy density of the electrons. When this condition is not fulfilled, the runaway electron current, if it is contained in the plasma and constrained to follow a closed circuit, will grow rapidly at the expense of the conduction current.

With regard to runaway ions, however, the situation is not quite so simple since the separate contributions to the dynamic friction from the ion-ion

and ion-electron encounters have their maximum effect at widely different values of the ion energy. The omission of the ion-electron contribution can lead to erroneous conclusions (GEORGE, 1958), as shown by GIBSON (1959). PARKER and TIDMAN have studied the formation of suprathermal ions in a number of instances and have also neglected the effect of the ion-electron encounters, although in this case their results are apparently not seriously affected by this omission. When the ion-electron encounters are taken into account it is evident that ions do not runaway as readily as the electrons. As an example, in present day pinch discharges deuterons require an energy approaching 0·5 MeV before they are finally runaway (THOMPSON *et al.*, 1958). The problem of how ions may be decoupled from a thermal state is of particular importance, for example, in pinched discharges (THONEMANN *et al.*, 1958; TUCK *et al.*, 1958; WARE *et al.*, 1958) in view of the possible existence of energetic deuterons (ROSE *et al.*, 1958, JONES *et al.*, 1958); and is also of importance in astrophysics, since cosmic rays and other particle fluxes of suprathermal energy most probably originate from a thermal state. Because the ion-electron contribution is so important, it is essential that the dynamic friction should take into account the modified electron velocity distribution when considering whether an applied electric field will accelerate an ion. Following this procedure, it is shown that ions can gain energy from the electric field when their charge is less than the mean ionic charge of the plasma. If the ion has a charge Z_i, and the mean ionic charge is Z, the ion is accelerated in the direction of the electric field until it attains a speed approximately equal to $Z/Z_i - 1$ times the mean drift speed of the electrons. Clearly, it is extremely unlikely that runaway ions will be produced in this manner. If, however, runaway electrons are not perfectly constrained to a closed circuit but are continually lost from regions in which they are generated and accelerated, then the ions in these regions may achieve appreciable momentum. Alternatively, suprathermal ions can be generated by the Fermi mechanism; in this case the ions are accelerated by collisions with magnetic fields embedded in systematic or disordered mass motions, as discussed by PARKER and TIDMAN. This subject is treated in a slightly more elementary manner, as compared with the investigations of these authors. The present treatment stresses the distinction between the Fermi interactions which are responsible for accelerating a large fraction of the ions, and therefore for heating the ion gas, and the Fermi interactions which affect only a small fraction of the ions and accelerate it to relatively large suprathermal energies. Here again the decision as to which of these two processes applies is governed by the ratio of the magnetic and ion thermal energy densities. It is possible that in the future accelerators may be constructed which are based on principles of this kind.

The subject of suprathermal particles is treated, first by discussing a uniform plasma in which the ions and electrons have Maxwellian velocity distributions (Section 2); secondly, in a current conducting plasma where the perturbed electron velocity distribution is taken into account (Section 3); the estimation of runaway electron currents and their possible effects in ionized gases is considered in Section 4; finally the subject of ion acceleration from a thermal state, primarily by Fermi interactions, is touched on in Section 5.

2. UNIFORM PLASMA

2.1 *Dynamic friction for an arbitrary test particle*

In a fully ionized gas, in which the ions and electrons have Maxwellian velocity distributions, the effect of collisions on a test particle can be found (SPITZER, 1956) from the results of CHANDRASEKHAR's (1942, 1943, 1949) investigations into the motion of a test star within a star field. Thus, a test particle of mass m_t and charge e_t, having a velocity v_t and moving in a plasma consisting of field particles of mass m_f, charge e_f, density n_f, and temperature T_f, experiences a mean rate of change of momentum in the direction of its velocity of

$$\langle m_t \Delta v_t \rangle^0 = -\sum \frac{D_{tf} n_f}{kT_f} \left(1 + \frac{m_f}{m_t}\right) G(x_f) \qquad (1)$$

as a result of the large number of relatively distant encounters. The zero superscript in this expression for the dynamic friction is used to indicate that the field particles have Maxwellian velocity distributions; also $x_f = v_t(m_f/2kT_f)^{1/2}$, $D_{tf} = 4\pi e_t^2 e_f^2 \ln \Lambda$, where Λ is the mean ratio of the Debye shielding distance to the impact parameter for large angle scattering (SPITZER, 1956) ; and

$$\begin{aligned} G(x) &= \{\Phi(x) - x\Phi'(x)\}/2x^2 \\ &\simeq 2x/3\pi^{1/2} \quad (x < 1) \\ &\simeq 1/2x^2 \quad (x > 1) \end{aligned} \qquad (2a,b,c)$$

where Φ is the error function. When $x_f < 1$ (that is, the test particle has a speed less than the most probable or mean random speed of the field particles of type f) the dynamic friction with the f type particles conforms to Stokes's law.

At distances greater than the Debye shielding distance the test particle excites collective oscillations

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(PINES and BOHM, 1952) when its velocity exceeds the mean random velocity of the electrons. In the present treatment the interest is primarily in the initial stage of the acceleration when the particle is decoupled from its thermal state; this effect will therefore be small and can be neglected.

2.2 *Electron test particle*

Let the plasma consist of electrons ($f = 1$) having a temperature T_1 and a density n_1 and ions ($f = 2$) having a temperature T_2 and density $n_2 = n_1/Z_f$, where Z_f is the ionic charge. For an electron the dynamic friction is

$$\langle m_1 \Delta v_1 \rangle^0 = -Dn\left\{2\frac{G(x_1)}{kT_1} + Z_t\frac{m_2}{m_1}\frac{G(x_2)}{kT_2}\right\} \quad (3)$$

from equation (1), where $n_1 = n$, and the quantity m_1/m_2 as compared with unity is ignored. From equation (2c) it is seen that the electron, if it is to run away, must have initially a speed exceeding the mean random speeds of both the electron and ion gases, or $x_1 > 1$, $x_2 > 1$. Under these conditions equation (3) becomes

$$\langle m_1 \Delta v_1 \rangle^0 = -Dn(1 + 2^{-1}Z)/kT_1 x_1^2 \quad (4)$$

where Z now stands for the mean ionic charge number defined by

$$Z = n_2^{-1}\Sigma n_i Z_i = n^{-1}\Sigma n_f Z_f^2 \quad (5)$$

when the ionized gas consists of ions of different charge.

If current is conducted, as the result of an electric field, the velocity distribution of the electrons is modified and equation (1) gives only a first approximation to the value of the dynamic friction. However, in Section 3 it will be seen that equation (4) is sufficiently accurate provided $x_1 \gg 1$. It follows that an electron of $x_1 > x_1(\xi)$ is continuously accelerated parallel to an electric field of

$$E = Dnc(1 + 2^{-1}Z)/ekT_1 x_1^2(\xi)$$
$$= 3 \times 10^7 n \ln\Lambda(1 + 2^{-1}Z)/Tx_1^2(\xi) \text{ volt cm}^{-1} \quad (6)$$

The dynamic friction, in terms of volts per cm, is shown in Fig. 1 for an electron moving through an electron–proton plasma in which $T_1 = T_2$ and $\ln\Lambda \simeq 13$

2.3 *Ion test particle*

For a test ion of charge Z_t the dynamic friction is

$$\langle m_2 \Delta v_2 \rangle^0 = -Z_t^2 Dn\left\{\frac{G(x_1)}{kT_1} + 2Z_f\frac{G(x_2)}{kT_2}\right\} \quad (7)$$

from equation (1). The friction has a first maximum when the ion velocity is equal in magnitude to the mean random speed of the ion gas, or $x_2 = 1$, as illustrated in Fig. 1. Thereafter, the friction drops to a minimum value

$$\langle m_2 \Delta v_2 \rangle^0_{\text{trough}} = -(Z_t^2 Dn/kT_1)(3Z_f m_1/\pi m_{2f})^{1/3} \quad (8)$$

when the ion has an energy of

$$(2^{-1}m_2 v_2^2)_{\text{trough}} = 3Z_f kT_1(\pi m_{2f}/3Z_f m_1)^{1/3} \quad (9)$$

These results for the trough in the dynamic friction curve are obtained by using the approximations of equations (2b,c), and their independence of the ion temperature holds provided $T_1/T_2 \ll m_{2f}/m_1$.

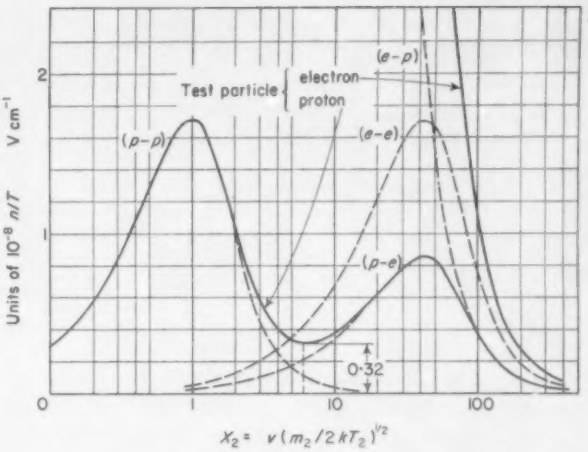

FIG. 1.—Dynamic friction curves for electron and proton test particle moving with a velocity v in a fully ionized gas in which $T_1 = T_2$ and $\ln\Lambda = 13$.

Between the first maximum and the trough minimum of the dynamic friction an ion is free to enter into an initial runaway state. Beyond the point of minimum friction the ion ceases to be runaway owing to the increasing contribution to the friction from the ion-electron encounters. The second maximum occurs when the ion attains a velocity equal in magnitude to the mean random speed of the electron gas, or $x_1 = 1$, corresponding to an ion energy of $m_2 kT_1/m_1$; this is only approximately true, for no allowance has been made for the effect of the accelerating field on the electron distribution. Beyond the second maximum the ion is free to enter into a second and final runaway state.

3. CURRENT CONDUCTING PLASMA

3.1 *An arbitrary test particle*

In the previous Section it was assumed, rather unrealistically, that the Maxwellian velocity distributions remained undisturbed by the presence of an electric field, or any other force field which might accelerate a given test particle.

TABLE 1.—THE FUNCTIONS $L_1(x)$ AND $L_4(x)$

x	0·1	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6
$L_1(x)$	0·018	0·070	0·262	0·526	0·800	1·033	1·196	1·284	1·311
$L_4(x)$	0·000	0·000	0·006	0·040	0·137	0·322	0·586	0·886	1·161
x	1·8	2·0	2·5	3·0	3·5	4	5	10	
$L_1(x)$	1·297	1·262	1·163	1·098	1·062	1·041	1·021	1·003	
$L_4(x)$	1·367	1·483	1·474	1·324	1·210	1·141	1·072	1·009	

To avoid considerable computation, the electron-electron encounters within the plasma are ignored so far as they contribute to the modified electron distribution. It turns out that the assumption of a Lorentz-type ionized gas is justified for the particular ranges of x_1 which are of primary interest. Following the work of COHEN *et al.* (1950) one finds, after some adaptations, that the dynamic friction in a weak electric field for an arbitrary test particle is

$$\langle m_t \Delta v_t \rangle = \langle m_t \Delta v_t \rangle^0 - E\frac{Z_t e_t}{Zc}\left(1+\frac{m_1}{m_t}\right)\{1-L_m(x_1)\}\cos\theta, \tag{10}$$

where the first term on the right hand side is given by equation (1), Z is the mean ionic charge number of equation (5), θ is the angle between $E e_t$ and v_t, and

$$L_m(x) = \{2x^{-3}K_{m3}(x) + K_{m0}(x)\}/K_{mn}(\infty),$$

$$K_{mn}(x) = \int_0^x y^{m+2}\exp(-y^2)\,dy,$$

in which m = 1 in a strong transverse magnetic field (SPITZER, 1952) and $m = 4$ when the transverse component of the magnetic field is zero. The functions $L_m(x)$ are tabulated in Table 1 and have maximum values of $L_1(x) \simeq 1{\cdot}3$ at $x \simeq 1{\cdot}6$, and $L_4(x) \simeq 1{\cdot}5$ at $x \simeq 2{\cdot}0$.

When the test particle has a velocity much greater in magnitude than the mean random speed of the electrons, then $x_1 \gg 1$ and $L_m(x)_1 \to 1$, and the dynamic friction tends to the same value as given by equation (1). When the velocity is much less in magnitude than the mean random electron speed, then $x_1 \ll 1$ and $L_m(x_1) = 0({x_1}^{m+1})$, and the dynamic friction is

$$\langle m_t \Delta v_t \rangle = \langle m_t \Delta v_t \rangle^0 - E\frac{Z_t e_t}{Zc}\left(1+\frac{m_1}{m_t}\right)\cos\theta \tag{11}$$

From the numerical values tabulated by SPITZER and HÄRM (1953) it is apparent that in the ranges $x_1 \gg 1$ and $x_1 \ll 1$ the electron-electron encounters are of negligible effect on the value of $L_m(x)$. Hence, provided the discussion is restricted to a consideration of the dynamic friction in these ranges of x_1 the assumption of a Lorentz-type plasma is justified.

3.2 *Electron test particle*

An electron, having a velocity much greater than the mean random speed of the electron gas, experiences a dynamic friction approximately the same as when the velocity of the electron gas is Maxwellian. The conditions for runaway are then very much the same as given previously in Section 3.

An approximate criterion for a weak electric field is that the electron drift velocity is small compared with the mean random speed of the electron gas. If the Lorentz resistivity is η_L (SPITZER, 1956), and the plasma has a resistivity of $\eta = \eta_L/\chi$, this criterion gives $E \ll Z\pi^{1/2}Dnc/16\chi ekT_1$. In a strong transverse magnetic field, in which there are no Hall currents, $\chi = 3\pi/32$; when the transverse field is zero $\chi \simeq 3\pi/16$ for $Z = 1$, and increases to $\chi = 1$ for $Z = \infty$. From equation (6) it follows that an electron only runs away for $x_1 > x(\xi)$, where

$${x_1}^2(\xi) \gg 16\chi(1/2 + 1/Z)/\pi^{1/2}.$$

The condition that the electric field is weak thus ensures that only a small fraction of the electrons in the high energy tail are in the process of running away.

3.3 *Ion test particle*

For an ion, having a velocity much greater in magnitude than the mean random speed of the electron gas, the dynamic friction is approximately the same as when the velocity distribution of the electrons is Maxwellian.

When the ion moves in the direction of the electric field at a velocity which is only a small fraction of the mean random speed of the electrons, the dynamic friction is

$$\langle m_2 \Delta v_2 \rangle = \langle m_2 \Delta v_2 \rangle^0 - E{Z_t}^2 e/Zc, \tag{12}$$

from equation (11). The sum of the forces acting on the ion, given by $F = EZ_t e/c + \langle m_2 \Delta v_2 \rangle$, is therefore

$$F = \langle m_2 \Delta v_2 \rangle^0 + EZ_t(e/c)(1 - Z_t/Z). \qquad (13)$$

It is seen that when the test ion has the same charge as the mean ionic charge of the plasma, the effective force is the dynamic friction of equation (7). In this particular case the ion behaves at low speeds as if it were in a plasma in which the electrons are in thermal equilibrium and in which there is no electric field. Hence, so far as the directed motion is concerned, the ion comes to rest relative to the ion gas, and therefore a weak electric field cannot decouple ions with $Z_t = Z$ from their thermal state. It is not until the ion attains a speed which is an appreciable fraction of the mean random speed of the electrons that the electric field contributes significantly to the effective force acting on the ion.

If the charge on the test ion is different from the mean ionic charge, the ion is accelerated until it reaches a terminal velocity which is largely determined by the ion-electron encounters. Setting $F = 0$ in equation (13), and using equation (7) with the condition $x_1 < 1$, the terminal or drift velocity of the ion is

$$v_T = -\bar{v}_1 \frac{3\pi}{32\chi}\left(\frac{Z}{Z_t} - 1\right) \qquad (14)$$

where $\bar{v}_1$ is the drift velocity of the electrons.

When the charge on the test ion is greater than the mean ionic charge (that is, $Z_t > Z$) the ion is accelerated and then drifts in the same direction as the electrons. This is because the electric field exerts a force proportional to Z_t, but the rate of momentum transfer from the electrons is proportional to Z_t^2. The minimum value of χ is $3\pi/32$ (strong transverse magnetic field), and the ions therefore drift at velocities which are always less than the electron drift velocity.

When the charge on the ion is less than the mean ionic charge (that is $Z_t < Z$) the ion is accelerated and then drifts in the direction of the electric field. It should be noticed that both the drift velocities v_T and $\bar{v}_1$ are measured with respect to the mean velocity of the ionic charge:

$$\bar{v}_2 = \sum n_f Z_f v_T / \sum n_f Z_f.$$

On using equation (14) and changing Z_t to Z_f, one finds that $\bar{v}_2 = 0$, and the ion current is zero.

Hughes and Kaufman (1959) have observed in a toroidal discharge that impurity ions, such as OV and CV, have drift velocities in the direction of the electric field. This observation is not inconsistent with equation (14), for the ionized gas as a whole might possess momentum due to the loss of energetic electrons striking the walls (Colgate *et al.*, 1958). As Ware (1959) has shown, the electrons need to lose in this way only a very small percentage of their momentum gained from the electric field for the ion gas to gain considerable momentum; in fact, the emission of runaway electrons might well produce a situation in which $\bar{v}_2 + \bar{v}_1 = 0$, and the current is carried by the positive ions.

In a pinched discharge in deuterium, heavily contaminated by impurity ions, the mean ionic charge might be as high as 2, but it is unlikely that it will be much higher. Using the minimum value of $3\pi/32$ for χ, and putting $Z = 2$, the deuteron drift velocity is $v_T = jc/ne \simeq Ic/Ne$, where I is the total current and N is the line density. For the values of I and N quoted by Bickerton and London (1958) for optimum conditions, $v_T = 2 \times 10^7$ cm sec^{-1} in Zeta, and $v_T = 1{\cdot}2 \times 10^7$ in Sceptre III. Measurements of the energy spectrum of the neutrons in Zeta (Rose *et al.*, 1958) and protons in Sceptre III (Jones *et al.*, 1958) reveal anisotropic distributions which can be explained on the basis that a small fraction of the deuterons is accelerated to energies of several keV in the direction of the electric field and makes collisions with stationary deuterons. Alternatively, the neutrons and protons are thermonuclear in their origin, since the ion temperatures are sufficient, and the deuteron gas has a mean velocity of 6×10^7 cm sec^{-1} in Zeta and 4×10^7 cm sec^{-1} in Sceptre III. Bearing in mind that the drift velocities computed above must be increased by adding the mean velocity of the ion gas as the result of runaway electrons, it is possible that the differences are not significant. However, this interpretation is not convincing without further information on the drift velocities of the ions.

The difficulty with the interpretation that energetic deuterons are in collision with almost stationary deuterons is to find a mechanism whereby a small fraction is accelerated to velocities an order of magnitude greater than their drift velocities (about 17 keV in Zeta); this point has been stressed by Gibson (1959). One possibility is that runaway electrons are generated and accelerated, not uniformly throughout the discharge, but preferentially in a restricted region; as, for example, close to the axis where the electric field in the direction of the magnetic field may have its greatest value. If these energetic electrons are continuously escaping, either to the walls or into the remainder of the discharge, the ions in the restricted region will experience considerable acceleration and could readily achieve the required energy.

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From WARE's computations one finds that the shift in the proton energy can be explained if the runaway electrons gain most of their momentum in a region containing approximately 10 per cent of the particles. Not necessarily 10 per cent of the deuterons will be accelerated, however, as many will diffuse out of the accelerating region.

4. RUNAWAY ELECTRON CURRENTS

4.1 *Rate of diffusion into the runaway state*

The rate at which thermal particles of a given kind diffuse into the runaway state can be determined approximately in the following manner. In an electric field particles of $v_t > v(\xi)$ will run away, where $v(\xi)$ is the surface in velocity space defined by

$$(e_t/c)E \cos\theta + \langle m_t \Delta v_t \rangle = 0.$$

The error is small if it is assumed that the $v(\xi)$ surface is a hemisphere whose axis is in the direction of $e_t E$. The Boltzmann equation for the runaway particles of type t is

$$\frac{\partial f_t^r}{\partial t} + \mathbf{v} \cdot \nabla f_t^r + \left(\frac{e_t}{m_t c}\right)\mathbf{E} \cdot \nabla_v f_t^r = \left(\frac{\partial f_t}{\partial t}\right)_{\text{coll}}, \quad (15)$$

where f_t^r is their distribution function. For the present purpose the collision term is related to the rate at which diffusion takes place across the $v(\xi)$ surface, and can be evaluated by supposing that $f_t = f_t^0$ is Maxwellian for $v \leqslant v(\xi)$. In this case the Fokker–Planck equation (CHANDRASEKHAR, 1943; 1949) for $v \leqslant v(\xi)$, is

$$\frac{\partial f_t^r}{\partial t} = \nabla_v \cdot (q \nabla_v f_t^0 + f_t^0 \langle \Delta \mathbf{v}_t \rangle),$$

where $q = \langle \Delta v_t \rangle kT/mv$ is a diffusion coefficient. By Gauss's theorem, the flux diffusing outwards across the hemispherical $v(\xi)$ surface is $2\pi v^2(\xi) q (\partial f_t^0/\partial v)_\xi$; but since f_t^0 is Maxwellian, this is also equal to $-2\pi v^2(\xi) f_t^0(\xi) \langle \Delta v_t \rangle_\xi$, where $m_t \langle \Delta v_t \rangle$ is given by equation (1). Therefore, integrating equation (15), one finds approximately that

$$\frac{\partial n_t^r}{\partial t} + \frac{c}{e_t} \nabla \cdot \mathbf{j}_t^r = -2\pi v^2(\xi) f_t^0(\xi) \langle \Delta v_t \rangle_\xi^0. \quad (16)$$

A slightly different result has previously been used (HARRISON, 1958). DREICER (to be published) finds that an expression of this form under-estimates the rate of generation of the runaway flux, primarily due to the assumption that the distribution is Maxwellian up to the $v(\xi)$ surface. The difference between equation (16) and the results computed by DREICER, however, is less pronounced than the difference with the previously published result (HARRISON, 1958).

4.2 *Runaway electron currents*

In the case of an electron runaway current of density of $j_1^r \equiv j_r$, equation (16) gives, with equation (4)

$$\frac{\partial n_r}{\partial t} + \frac{c}{e} \nabla \cdot \mathbf{j}_r = \frac{4Dn^2}{(2kT_1)^{3/2}(m_1\pi)^{1/2}} (1 + 2^{-1}Z) \exp -x_1^2(\xi),$$

$$= 32 \frac{n^2 \eta_L a_1}{\pi} (1/2 + 1/Z) \exp -x_1^2(\xi), \quad (17a,b$$

where η_L is the Lorentz resistivity and $a_1 = e^2/m_1 c^2$ is the classical electron radius. As a first approximation it is assumed that quasi-neutrality is preserved in the thermal state.

The effect of runaway electrons can be regarded in one sense as a reduction in the resistivity of the plasma But as COWLING (1956) has pointed out in connexion with runaway fluxes in solar flares, current in most cases is not limited by resistance but by the self-inductance of the system. In this paper we shall adopt a rather simplified model in which there are no dynamo effects and the magnetic field is maintained by currents in the plasma which are decaying due to Joule dissipation. In astrophysical applications such a model is not unrealistic, although the time of decay for the magnetic field is usually very large; in laboratory discharges, however, the time of decay is small, usually only a fraction of a second, and such a model will be unrealistic when the currents are maintained by external means. Under these circumstances a runaway current, for example, could never consist of all the original conduction electrons as this would mean that their final mean velocity was merely that of the initial drift velocity. What must happen is that the runaway current, provided it is constrained to follow a closed circuit, will grow at the expense of the conduction current, and given sufficient time, the electric field will eventually be reduced to a small value consistent with the rate of momentum transfer from the fast electron fraction. The resistivity of the plasma and the distribution of the magnetic fields will determine to what extent and in what places the runaway current will supplant the conduction current of the local surrounding regions.

On the basis of the assumption that all runaway electrons are constrained to flow in a closed circuit, the following cases are considered using an order of magnitude treatment: (i) a widely distributed runaway current; (ii) the formation of runaway beams on the neutral lines of a transverse magnetic field; and

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(iii) a 'super conducting' plasma which excludes an applied electromagnetic field. The problem of stability is not dealt with, and it is therefore emphasized that the results in many cases may be totally unrealistic.

4.3 *A widely distributed runaway current*

The plasma is considered to be quiescent, and free from mass motions and collective interactions. We now ask: what are the conditions that after a time τ_r the runaway current is everywhere very much less than the conduction current?

To answer this question we must consider whether the runaway electrons become relativistic in a time small compared with τ_r. The time taken for an electron to become relativistic is given by (using electromagnetic units and the electronic charge equal to e/c)

$$t \simeq m_1 c^2/eE \tag{18}$$

In the hydromagnetic equation (COWLING, 1957)

$$\partial \mathbf{H}/\partial t = \nabla \times (\mathbf{v} \times \mathbf{H}) + (\eta/4\pi)\nabla^2 \mathbf{H}$$

the magnetic field diffuses or decays in a time of the order

$$\tau_H \simeq 4\pi L^2/\eta \tag{19}$$

where L is a characteristic length in which the field H changes by a relatively large amount. Now $\tau_r \leqslant \tau_H$, and the condition that the runaway electrons are largely relativistic is therefore: $j_c \gg m_1 c^2/4\pi e L^2$, where j_c is the conduction current density. In a discharge of radius R, the electron line density is $N \simeq \pi R^2 n \sim \pi L^2 n$, and using the parameter $\nu = Na_1$, this condition gives

$$\bar{v}_1 \gg c/4\nu, \tag{20}$$

for the electron drift velocity. When the condition (20) holds, the runaway current density is approximately $j_r = n_r e$.

Before proceeding further, the assumption that the electric field is weak, and therefore the conduction current is $j_c = E/\eta$, will be examined. The current density is $j_c = ne\bar{v}_1/c \sim H/4\pi L$; and if, as before, $N \sim \pi L^2 n$, and also

$$\beta_f = \frac{n_f k T_f}{H^2/8\pi}, \tag{21}$$

it follows that for $\bar{v}_1 = \bar{x}_1(2kT_1/m_1)^{1/2}$:

$$\bar{x}_1 \beta_1 \nu = 1/4 \tag{22}$$

In a weak electric field the electrons drift with a velocity which is small compared with their mean random speed, or $\bar{x}_1 \ll 1$, and therefore $\beta_1 \nu \gg 1/4$. LAWSON (1957; 1958) has deduced a similar relation for a current conducting plasma, given by $\nu \gg 1/2$.

Non-relativistic runaway electrons. On integrating equation (17b) one finds (HARRISON, 1958) for non-relativistic electrons with div $\mathbf{j}_r = 0$, that

$$j_r = (4\tau_r n a_1)^2 (\eta_L E/\pi)(1/2 + 1/Z) \exp -x_1^2(\xi), \tag{23}$$

where E and n are taken to be constant on the assumption that $j_r \ll j_c$. Using the hydromagnetic time constant τ_H, and a resistivity of $\eta = 16\eta_L/3\pi$, it follows

$$\frac{j_r}{j_c} = 48 \left(\frac{\tau_r}{\tau_H}\right)^2 \nu^2 (1/2 + 1/Z) \exp -x_1^2(\xi). \tag{24}$$

The value of $x_1^2(\xi)$ can be estimated from equation (6), and is

$$x_1^2(\xi) \simeq 32(na_1\beta_1)^{1/2} \chi L(1/2 + 1/Z),$$
$$\simeq 6(\pi\nu\beta_1)^{1/2} (1/2 + 1/Z). \tag{25a,b}$$

The condition that in a time $\tau_r = \tau_H$ the runaway current shall be small in comparison with the conduction current, is given by

$$9(\pi\nu\beta_1)^{1/2} > \ln (72\nu^2), \tag{26}$$

for $Z = 1$. This is satisfied for all values of ν when β_1 is of the order unity. Alternatively, using equation (22), the above condition becomes

$$\bar{x}_1 \leqslant 9\pi^{1/2}/2 \ln 72\nu^2 \tag{27}$$

In many discharges $\nu \sim 10^4$, and if the runaway current is to approach anywhere near the value of the conduction current, it follows that $\beta_1 \sim 1/5000$. The runaway electrons will tend to follow the lines of force of the magnetic field, and this must imply a configuration that tends to be force-free in which the gas pressure is small in comparison with the magnetic pressures.

Relativistic runaway electrons. When the electrons become rapidly relativistic, $j_r = n_r e$, and from equation (17b)

$$j_r = 32(\tau_r n^2 a_1 e \eta_L/\pi)(1/2 + 1/Z) \exp -x_1^2(\xi). \tag{28}$$

Using the same assumptions as before one finds that

$$\frac{j_r}{j_c} \leqslant 96\, \nu^2 \left(\frac{\tau_r}{\tau_H}\right) (1/2 + 1/Z) \exp - x_1^2(\xi), \tag{29}$$

where $x_1^2(\xi)$ is given by equation (25b). However, the relativistic runaway current will be less than the non-relativistic runaway current given by equation (24) and we shall therefore use equation (24) as giving the maximum current according to the present treatment.

It would appear from the conditions (26) and (29) that in a containment time smaller or equal to the

characteristic life-time τ_H, the runaway current will be relatively small when $\nu\beta_1 \gg 1$, For a discharge of $\nu = 10^4$, condition (26) requires that $\beta_1 > 1/5000$; it may be noticed that if the currents of $I \simeq 10^5$ A in present day stabilized discharges are increased by a factor of 20–30, with only small alterations in N and T_1, then the runaway current will no longer necessarily be small in comparison with the conduction current. In this case the runaway current would cease to be widely distributed but would tend to concentrate at the surface.

4.4 *Runaway electron beams*

When the magnetic field is mainly transverse to the direction of the electric current, the electrons will be prevented from running away except along neutral lines of the magnetic field. A runaway beam therefore may develop in the region of a neutral line at the expense of the total conduction current, and thus form an 'inverse skin effect' in the current distribution.

A beam of this kind must form in a time equal to or less than τ_H, which is the characteristic time for the magnetic field to diffuse in on to the neutral line. The minimum condition for the runaway beam to grow at the expense of the total conduction current is therefore

$$18(\pi\beta_1\nu)^{1/2} < \ln(36\nu^2), \tag{30}$$

using equations (23) and (25b) with the modification that $\chi = 3\pi/32$. Within the runaway beam the electromagnetic field is of the form

$$\nabla^2\mathbf{H} = 4\pi n a_1 \mathbf{H}$$

where $l = (4\pi n a_1)^{-1/2}$ is the collision-free penetration depth. Hence, if the radius of the initial conduction current channel is $R \sim L \simeq (N_1/\pi n)^{1/2}$, and the radius of the beam formed on the neutral line is $r \sim l = (4\pi n a_1)^{-1/2}$, then

$$R/r \sim 2\nu^{1/2} \tag{31}$$

In a field which is largely transverse, the small value of β_1 indicated by the condition (30) implies that the magnetic pressure exerted on the gas is large compared with the gas pressure, and therefore if such beams of energetic electrons can occur, one of their necessary features is that they are developed in extremely non-steady plasmas, as for example, the solar corona (KIEPENHEUER, 1956) and possibly such regions as the Crab nebula (OORT and WALRAVEN, 1956). Obviously, electrons will run away in the region of a neutral line, even when the condition (30) is not satisfied, but their current will remain small compared with the total conduction current.

4.5 *Superconduction in plasmas*

The time taken for a magnetic field to diffuse into a plasma a distance L, equal to the collision-free penetration depth l, is $\tau_r \sim 4\pi l^2/\eta$. If in this time, the conduction current predominates over the runaway current, the plasma behaves as an ordinary fluid conductor. However, if the runaway current rapidly grows and predominates in this short time, the electromagnetic field is screened from the interior, and the plasma behaves as a fluid superconductor. The minimum condition for this, from equations (23) and (25a), is that $\beta^{1/2} \ll 16 \ln(3/2)/9\pi^{1/2}$, or

$$\frac{H^2/8\pi}{n_1 k T_1} \gg (9\pi^{1/2}/16 \ln 3/2)^2, \tag{32}$$

where $(9\pi^{1/2}/16 \ln 3/2)^2 \simeq 6$. When an external magnetic field exerts a pressure greater than the electron pressure by an order of magnitude or more, the runaway current predominates. The duration of this current must depend on a number of considerations; furthermore, if the duration is long as compared with the 'slowing down' time of the electrons (SPITZER, 1956), the magnetic field will penetrate further and the runaway effect may be regarded merely as an enhancement of the conductivity.

A plasma super-conductor differs from an ordinary super-conductor in that large electric fields are required to establish the super-currents, and therefore magnetic fields may exist in the interior of the plasma. These interior magnetic fields will form super-currents on the surface as the plasma collapses under the pressure of the external magnetic field, and therefore there is the possibility of an equilibrium configuration.

A plasma super-conductor resembles, however, an ordinary super-conductor in that the super-current has a maximum possible value; when this value is exceeded the magnetic field penetrates to greater depths and the plasma reverts to the state of a fluid conductor. The maximum possible intensity of the supercurrent sheet is nel (the electrons of density n move at the velocity c); and therefore $\beta_1 = 2kT_1/m_1c^2$. Thus, the condition for superconduction is

$$\{16 \ln(3/2)/9\pi^{1/2}\}^2 \gg \beta_1 \geqslant 2kT_1/m_1c^2. \tag{33}$$

It is interesting to notice that the super-current has its maximum value when the Alfvén wave velocity ($v_H = H/(4\pi n m_2)^{1/2}$) has the value

$$v_H = c(m_1/m_2)^{1/2},$$

and that the maximum possible total super-current $I_{\max}$ in a cylindrical plasma is given by

$$I_{\max}^2 = N m_1 c^2 \tag{34}$$

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The two stream collective Coulomb interaction operates when the wavelength $2\pi v_1/\omega_1$ is less than l, where ω_1 is the plasma electron frequency. Hence, when $v_1 > c/2\pi$ one expects that instabilities from collective interaction will be relatively unimportant.

5. SUPRATHERMAL IONS

5.1 *Acceleration by electric fields*

It has been seen that in a weak electric field ions are only accelerated by the field when their charge is less than the mean ionic charge. The speed achieved by these ions will rarely be very much greater than the drift speed of the electrons and as this is very much less than the mean random speed of the electrons, it is extremely unlikely that runaway ions will be produced in such fields. The acceleration of ions in conducting plasmas by a betatron mechanism, as proposed by SWANN (1933), ALFVÉN (1937), RIDDIFORD and BUTLER (1952), is unlikely unless the ions obtain initially a sufficiently high energy from some other source; similarly, KIEPENHEUER'S (1950; 1958) suggestions for acceleration in the direction of electric fields, and the use of slowly varying electromagnetic fields (SALISBURY and MENZEL, 1948) are subject to the same initial difficulty.

In a strong electric field the most pronounced effect is that the electrons run away. If the electrons are constrained to follow a closed circuit, the local electric field will diminish, and the eventual effect on the ions must be small; the ion gas as a whole will possess a small amount of momentum equal and opposite to that of the electrons. When, however, the runaway electrons are not perfectly constrained, but are continually lost either partly or completely from the regions in which they are generated, the ions in these regions will be accelerated. This type of acceleration may be effective even when the electric field is weak, as previously pointed out; and although this mechanism might produce suprathermal ions having energies within the range kT_2 to m_2kT_1/m_1 it seems unlikely, at least in astrophysical applications, that appreciable high energy fluxes of runaway ions will be generated.

5.2 *Fermi-type acceleration*

In the mechanism proposed by FERMI (1949; 1954) cosmic ray particles gain energy by collisions with magnetic fields embedded in moving gas clouds. For a non-relativistic particle it is easily seen that the mean rate of increase in speed for head-on collisions with a magnetic field moving at a velocity u is $\langle \Delta v \rangle_F = 2u^2/\lambda_H$, where λ_H is the mean free path between such collisions. PARKER and TIDMAN (1958a) have considered in more detail the idealized case where the turbulent field bearing elements are spherical and have a mean speed of u, and find for random collisions that $\langle \Delta v \rangle_F = u^2/\lambda_H$. The coefficient of u^2/λ_H would appear to be of the order of unity even when the turbulent elements are not spherical. It should be pointed out that systematic and periodic mass motions can also provide similar or even greater rates of acceleration. Several authors (DAVIS, 1956; PARKER, 1958b) have found mean energy gains per collision of the first order in u/c at relativistic energies, in the non-relativistic region this is equivalent to $\langle \Delta v \rangle_F \simeq vu/\lambda_H$.

Ignoring the more efficient of the two mechanisms, a crude criterion for accelerating a given ion is that the mean rate at which momentum is gained from the Fermi interactions must be greater than the mean rate of momentum loss due to Coulomb interactions, or

$$\langle m_2 \Delta v_2 \rangle_F + \langle m_2 \Delta v_2 \rangle > 0 \qquad (35)$$

If $\lambda(v)$ is the mean free path of the ion, this gives

$$u^2/\lambda_H - v_2^2/\lambda(v) > 0.$$

Acceleration will not occur unless $\lambda(v) \geqslant \lambda_H$, where $\lambda(v)$ increases with velocity, and therefore particles of $v \geqslant v(\xi)$ are accelerated provided $v(\xi) > u$. From equation (35) it follows that the minimum value of $v(\xi)$ is given by

$$v^4(\xi) = 2Z_i^2 ZDn\lambda_H/m_2^2, \qquad (36)$$

where only ion-ion encounters are considered. If u is of the same order as the ALFVÉN wave velocity ($u \simeq H/(4\pi n_2 m_2)^{1/2}$), then $\beta_2 x_2^2(\xi) \simeq 1$. By taking the minimum value of $v(\xi)$, the number of ions affected by the Fermi interactions is overestimated. On comparing the Fokker–Planck equations for the Fermi and Coulomb interactions PARKER (1958c) has deduced a value of $v(\xi)$ which is even smaller, by about $\beta_2^{1/4}$, or more exactly $(4\beta_2/3)^{1/2} x_2^2(\xi) = 1$. The difference is perhaps not so important in view of the extreme approximations made in estimating the rate at which suprathermal ions are formed.

One notices that when β_2 is close to unity, as for example in the BENNETT (1934) pinch relation where $\beta_2 \simeq 1/2$, all the ions with speeds somewhat greater than the mean random speed are subject to Fermi interactions. This cannot mean, however, that the ion gas diffuses rapidly into a state of continuous acceleration, as the energy densities of mass and thermal motion are about equal. The energy in the disordered motions in this case leads merely to an increase in the ion temperature. If the turbulence is maintained by an external energy source, or by

instabilities of limited growth such as may be responsible in part for the large field mixing in Zeta (THOMPSON *et al.*, 1958), the ion gas will be rapidly heated by the Fermi interactions with the disordered motions of the magnetic field, as first suggested by PARKER and TIDMAN (1958). Hence, when β_2 is close to unity the mass motions will heat the gas; however when, β_2 is small, only a minute fraction of the ions is accelerated, and these will achieve relatively high energies.

A rough estimate of the difference between the ion and electron temperatures is given by

$$\frac{3}{2}k\frac{T_2 - T_1}{t_{12}} \simeq \frac{m_2 u^2}{\lambda_H}\int_u^\infty 4\pi v^3 f_2^0\, dv,$$

where t_{12} is the relaxation time for energy exchange between the ion and electron gases, as given by SPITZER (1956). Thus, in a discharge such as Zeta, with $n = 10^{14}$, $\beta_2 \simeq 0{\cdot}25$, $\lambda_H \simeq 10$ cm, one finds, for example, that when the deuterons have a temperature of $T_2 = 10^6$ °C, the electron temperature is $T_1 = 10^5$ °C.

5.3 *Rate of formation of suprathermal ions*

By following the same procedure as in Section 4.1 and observing that the $v(\xi)$ surface is now spherical, the rate of generation of suprathermal ions is again given by equation (16), provided one makes the approximation that the probability of acceleration is 1/2 for every ion diffusing across the $v(\xi)$ surface. For most ions in the high energy tail the dynamic friction comes from the ion-ion encounters, and therefore neglecting the ion-electron contribution

$$\frac{\partial n_r}{\partial t} = \{4Z_t^2 Z Dn^2/(2kT_2)^{3/2}(\pi m_2)^{1/2}\}\exp -x_2^2(\xi),$$
$$= 32Z_t^2 n^2 \eta_L a_1 (m_1/\pi^2 m_2)^{1/2}\exp -x_2^2(\xi). \qquad (37a,b)$$

Equation (37a) is identical with that obtained by PARKER and TIDMAN (1958a) from the diffusion computations of MACDONALD *et al.*, (1957).

In a time τ_H, the fractional number of suprathermal ions generated in a discharge with a line density of $N \sim \pi L^2 n$, is

$$n_r/n \sim 12\nu(m_1/m_2)^{1/2}\exp(-1/\beta_2), \qquad (38)$$

for $Z_t = 1$, $\chi = 3\pi/32$, and $x_2^2(\xi) = 1/\beta_2$. PARKER's smaller value of $x_2^2(\xi)$ gives a greater rate of generation. This equation emphasizes that unless either β_2 or the containment time of the suprathermal ions is relatively small, the energetic fraction will become impossibly large for $\nu \sim 10^4$. It has been suggested by BURKHARDT and LOVBERG (1958) and TUCK *et al.* (1958) that the nuclear reactions in pinched discharges in deuterium may possibly be caused by deuterons accelerated by Fermi interactions. It is seen that the energetic ions accelerated in this manner will be focused towards the axis, and therefore preferentially contained, when moving in the direction of the electric field, (THOMPSON, unpublished), and this could account for the anisotropy of the energetic distributions of the neutrons and protons. The difficulty with this suggestion is the apparently small values of β_2 which are required in order that only a relatively few of the deuterons are accelerated to the required energy.

If an ion is to be continuously accelerated and eventually become runaway, the Fermi rate of gain of momentum must be at least equal to the maximum value of the ion-electron dynamic friction, or

$$m_2 u^2/\lambda_H - Z_t^2 DnG(1)/kT_1 \geqslant 0, \qquad (39)$$

where $G(1) = 0{\cdot}214$. Using the ALFVÉN wave velocity for u, this gives

$$\lambda_H \beta_2 \leqslant 2k^2 T_1 T_2/Z_t^2 DnG(1),$$
$$\leqslant 2 \times 10^4\, T_1 T_2/n \qquad (40)$$

for $Z_t = 1$ and $\ln \Lambda \simeq 13$.

A solar flare provides an example of a turbulent plasma in which one might expect $\beta_2 \ll 1$, thus giving an emission of high energy ions. As an illustration, in the solar flare of February 26, 1956 (MEYER *et al.*, 1956; SIMPSON, 1958) it has been estimated by PARKER (1957) that the total number of high energy protons generated was of the order 10^{33}. From equation (37a) the rate at which suprathermal protons are produced per unit volume is

$$\partial n_r/\partial t \simeq 3{\cdot}5(n^2/T^{3/2})\exp(-1/\beta_2),$$

for $Z_t = 1$ and $\ln \Lambda \simeq 13$. Taking $T_1 = T_2 = 10^5$ °C, $n = 10^{11}$ cm^{-3}, and a flare volume of the order $(10^9)^3$ cm^3, then $\beta_2 \simeq 1/20$ if the protons are decoupled from their thermal state in 1 sec. The small scale turbulence of $\lambda_H < 4 \times 10^4$ cm in a magnetic field of approximately 30 G is therefore primarily responsible for decoupling the protons from the gas. Once the protons become suprathermal they are free to be accelerated by the grosser motions of the flare in which $\lambda < 10^9$ cm.

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EXPERIMENTS WITH PLASMA RINGS

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Abstract—The construction of a coaxial plasma gun is described. At its output end the gun is provided with a radial magnetic field, which is trapped by the plasma.

The plasma which is shot out from the gun is studied by photographic and magnetic methods. It is demonstrated that the gun produces magnetized plasma rings with the same basic structure as the rings obtained in toroidal pinch experiments.

When the the plasma rings are formed, the magnetic field lines from the gun break, a result which is of interest from a theoretical point of view.

1. INTRODUCTION

IN order to heat a plasma to a very high temperature it is convenient to give it a toroidal shape. In Zeta and similar machines, the plasma has a longitudinal magnetic field and is wrapped up in a poloidal field. After the contraction the larger parts of both fields are generated by currents in the plasma itself. Although there are reasons to suppose that this configuration should have a certain degree of stability, experiments show that the energy which is transferred to the plasma disappears very rapidly in a way which is not understood very well. There are two alternatives.

(1) The mechanism by which the plasma loses most of its energy is intimately connected with the structure of the plasma and the plasma will cool rapidly independently of the way in which the plasma is heated. This would be the case if, for example, there is a fundamental impossibility in keeping a plasma confined by a magnetic field generated by currents in the plasma itself.

(2) The rapid cooling is a consequence of the special way in which the plasma is heated. For example in the common machines the energy is transferred to the plasma by inducing a strong ring current in the plasma, and this produces runaway electrons, which may be harmful.

As all theories of the stability are worked out under idealized conditions which are difficult to realize experimentally, the best way of approaching this question seems to be by producing the same plasma configuration in a fundamentally different way.

This can possibly be done by means of a plasma gun, which shoots out magnetized plasma rings which are later brought to rest in a vessel where their stability can be investigated. This paper describes a method of producing such magnetized plasma rings. Experiments with the aim of trapping the rings and bringing them to rest have been started but will not be described here.

Plasma rings with a magnetization of the same type as in the Zeta and similar machines can be produced by a method outlined earlier by one of us (ALFVÉN, 1958). A magnetized plasma is produced by a discharge between coaxial electrodes and is shot out from them (Fig. 1a). Before it leaves the electrodes, it has to pass a radial magnetic field between a magnetic pole N in the inner electrode and another pole S, which is ring-shaped, in the outer electrode. When the plasma passes this field it takes the lines of force with it (Fig. 1b) provided that the conductivity and the density of the plasma are high enough. The lines of force will be stretched out and finally they can be expected to 'break' so that a free plasma ring is produced (Fig. 1c) which moves on. The ring should have the same magnetic configuration as in Zeta and similar machines.

In order to see whether a magnetized plasma ring could be produced in this way experiments were started in the Spring of 1958, and an apparatus was built which will be described here. Another somewhat modified apparatus was built and put into operation in the begining of 1959. The behaviour of the plasma, especially the formation of magnetized plasma rings, has been studied by means of electromagnetic probes and by photographic methods.

2. THE APPARATUS

The main features of the first of the devices are shown in Fig. 2. Basically the function is as follows: The current from a condenser bank C, is triggered by a spark gap G, and produces a discharge in the gas between the coaxial electrodes. This system will be referred to as the plasma gun. The material of the

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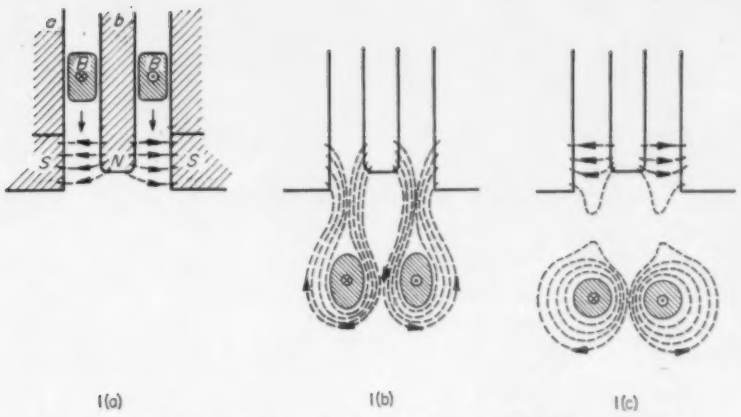

FIG. 1.—Capture of a plasma in a magnetic field.

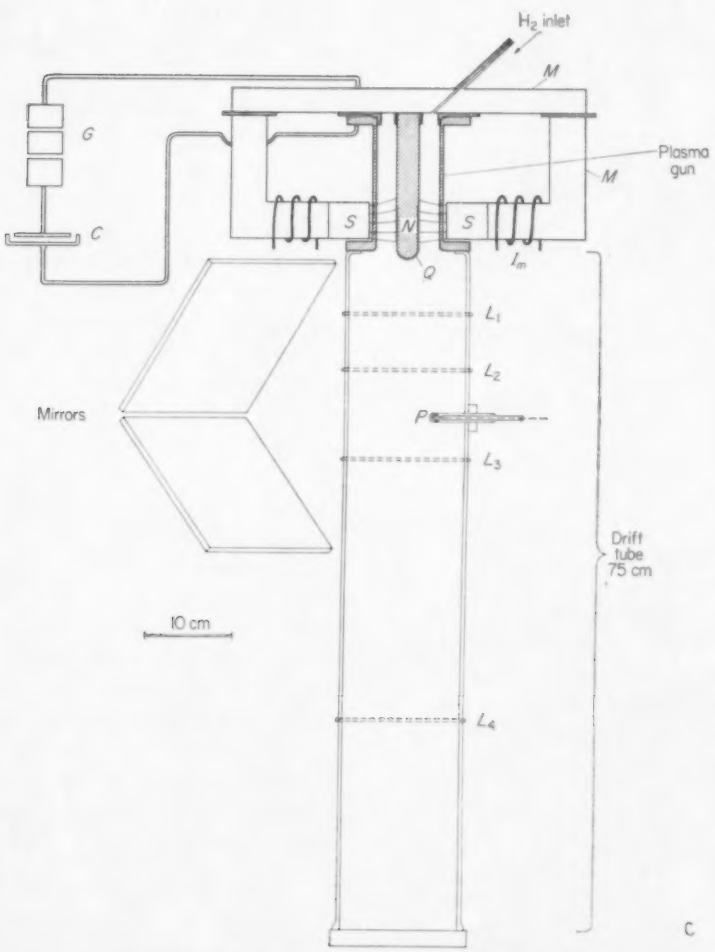

FIG. 2.—The plasma gun with drift tube and loops for magnetic measurements.

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centre electrode is iron, while the outer electrode is made of stainless steel.

By keeping the inductance low in the circuit the peak discharge current can be kept high even with a rather small condenser bank. In this apparatus a condenser bank of 190 μF, charged to 6 kV, yields a peak discharge current of approximately 250 kA. This is sufficient to provide a high degree of ionization.

The current is measured by an oscilloscope with the aid of a calibrated pick-up coil and an integrating RC-circuit ($R = 10^4\ \Omega$, $C = 0{\cdot}05\ \mu$F). By taking an appropriate signal from a divider, oscillograms are also obtained for the voltage across the electrodes in the gun. Typical current and voltage oscillograms are shown in Fig. 3. The oscilloscopes are triggered from another pick-up coil at the top of the gun, which measures the derivative of the discharge current.

The gas is hydrogen and the pressure usually 100 μ of Hg. A palladium valve at the H_2-inlet purifies the hydrogen. The pumps are in constant operation and purified hydrogen continually supplied.

Fig. 4 shows that the plasma which is produced in the gun is subject to an electrodynamical force, directed downwards. This force is strong enough to eject the plasma out of the gun down into a drift tube which consists of a 75 cm long glass tube, as shown in Fig. 2.

N and S are the poles of an air gap in a magnetic circuit M, magnetized by a constant current I_m. This stationary field is directed radially in the spacing between the coaxial electrodes. Q is a quartz tube, covering the lower part of the centre electrode, and ensuring that the discharge ignites above the region of the magnetic field.

After firing, the plasma leaves the gun and enters the drift tube roughly in the shape of a hollow cylinder. The velocity of the plasma is then approximately 6 cm/μsec, a value that seems fairly well preserved even in the lower part of the tube. This velocity seems to be remarkably constant even if the magnetic field, the electrode voltage or the gas pressure is varied within rather wide limits.

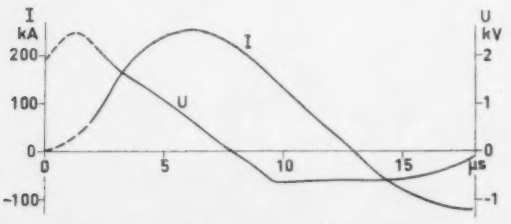

FIG. 3.—Oscillograms of the current I through the plasma gun and the voltage U between its electrodes (measured at the top of the gun).

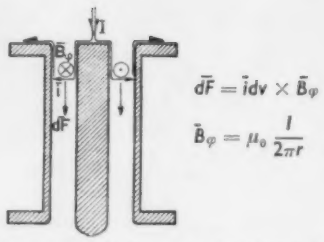

FIG. 4.—Electromagnetic forces on the plasma in the gun.

3. METHODS OF MEASUREMENT

For photographic study of the plasma two Kerr cell cameras are employed, which are triggered with the same pulse as the oscilloscopes. By means of a delay device the cameras can be set to release at arbitrary moments. The exact time of release is recorded on the current oscillogram and is seen there as a narrow spike. The time of exposure is 0·03 μsec. Because of the intense light conventional photographic material and processing can be used in spite of this short exposure.

For measurements of the magnetic field, probes P in the form of small coils with 10 turns and 4 mm diameter, have been introduced into the drift tube. Glass tubes 6–8 mm wide encase the probes and by use of O-ring seals they can be moved radially in the drift tube without disturbing the vacuum. Signals from the probes are displayed on oscilloscopes after integration in the same kind of circuits as described above for the plasma gun. The results are difficult to interpret, probably because the magnetic lines of force tend to be frozen in the plasma, so that the fields of the plasma do not penetrate into the coils. Furthermore the probes disturb the plasma. For these reasons the measurements with this method have been discontinued.

Instead the probes have been replaced by loops L, surrounding the drift tube. The integrated signal from such a loop is proportional to the instantaneous net flux through the loop or, which is the same, the back flow of magnetic flux outside the loop.

This flux should be compared with the 'static flux' by which we mean the magnetic flux in the plasma gun between the central rod and the ring-shaped pole of the magnet. This is measured with a fluxmeter and a probe coil around the central electrode at the top of the gun.

4. MAIN PROPERTIES OF THE DISCHARGE

When the radial magnetic field is zero, the plasma leaving the gun has a conical shape as shown by Figs. 5 (a) and 5 (b), which are taken at the same shot with 1·3 μsec time difference. To the left in these pictures

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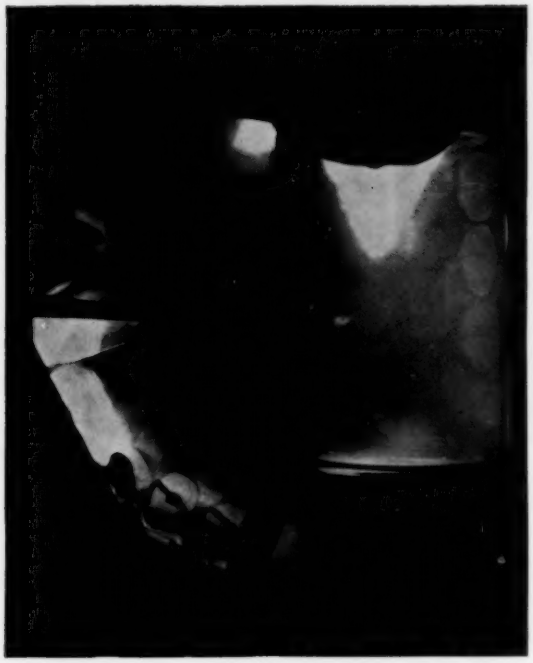


(a)

(b)

FIG. 5.—Plasma leaving gun when no static magnetic field is applied.

(a) 3·4 μsec
(b) 4·7 μsec
} after ignition.

To the left the plasma is seen through mirrors looking downwards and upwards. Drift tube only 22 cm long in this experiment.

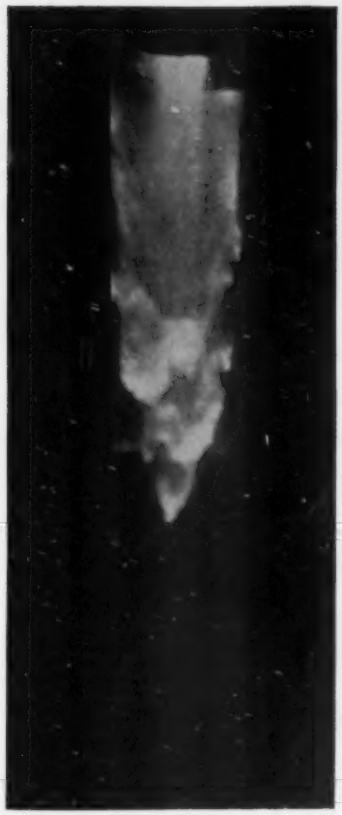
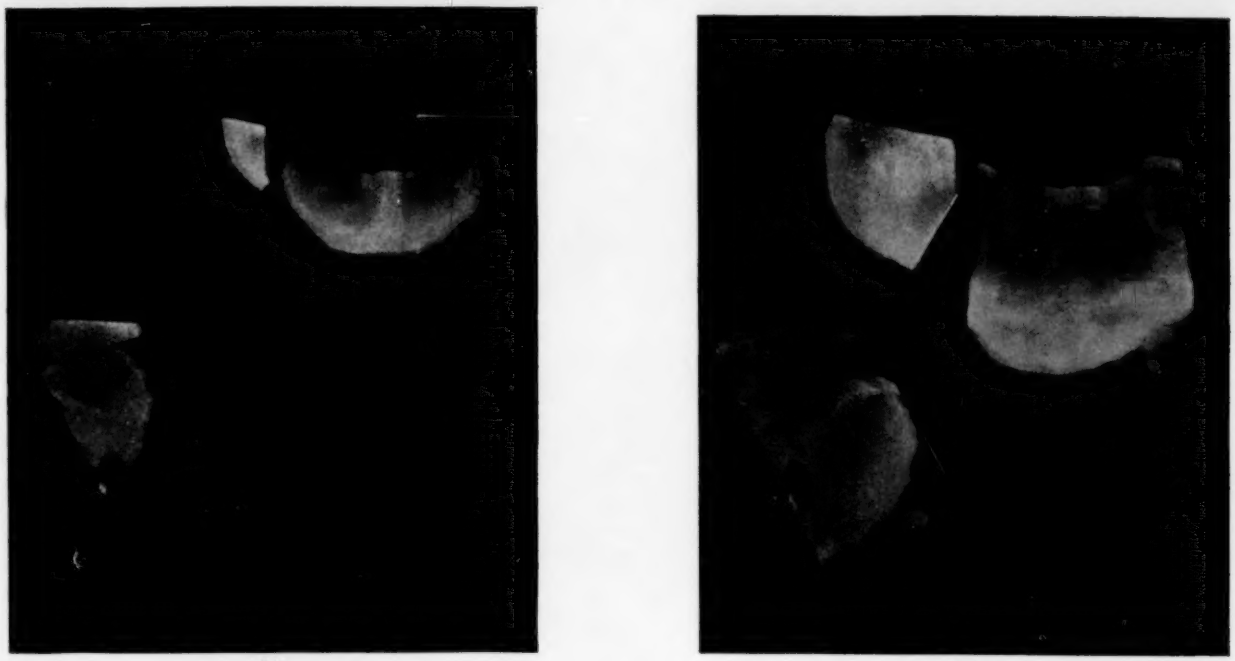


FIG. 6.—Unmagnetized plasma having travelled further into the drift tube. 10·7 μsec after ignition.



(a) (b)

FIG. 7.—Plasma magnetized by a strong static magnetic field ($\phi_s = 5{\cdot}5 \times 10^{-4}$ Wb)
(a) 4·2 μsec
(b) 6·3 μsec
after ignition.

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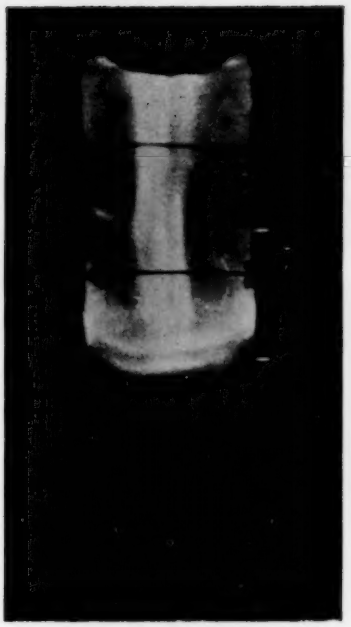

FIG. 8.—Magnetized plasma obtained with central electrode negative.

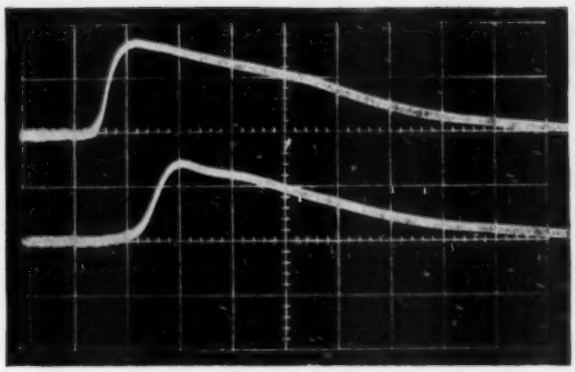

FIG. 9.—Flux measured by loops around the drift tube at 7 and 17 cm from the gun as function of time. Static magnetic flux $9{\cdot}5 \times 10^{-4}$ Wb. Flux scale 5×10^{-4} Wb/div. Time scale 2 μsec/div.

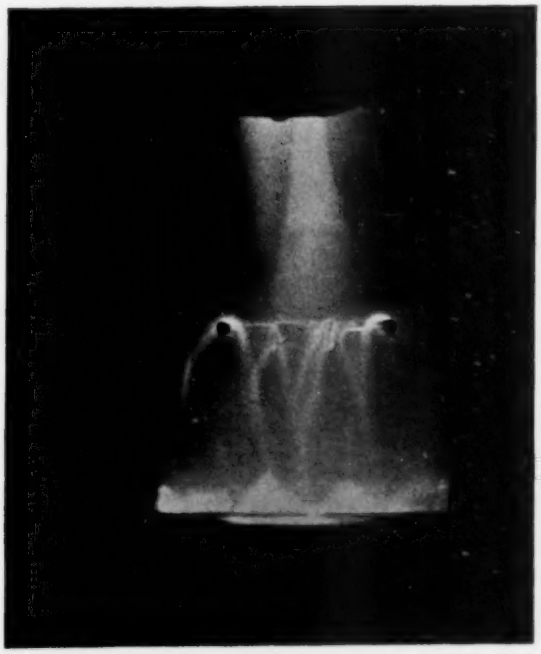

FIG. 15.—Plasma colliding with physical obstacles. In the middle of the picture four obstacles are seen: from left to right a glass rod, a thin copper sheet with edge pointing upwards, a mica sheet of similar shape and a copper rod. The picture is taken after the plasma front has reached the bottom of the drift tube.

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the discharge is seen from the side with the aid of two inclined mirrors (compare Fig. 2). The conical shape of the plasma front is maintained during the whole passage of the glass tube. In some cases, however, the plasma breaks up in a number of 'fingers' when it reaches the lowest part of the drift tube. An example is shown in Fig. 6.

for three values of the static flux. The flux oscillograms in each figure are collected from different shots and some of the irregularities seem to be due to statistical variations in the ignition of the discharge. In Figs. 10 and 11 the maxima of the flux measured by the last loop (at $z = 52$ cm) exceeds by a factor of two the flux measured by any other loop at the same instant.

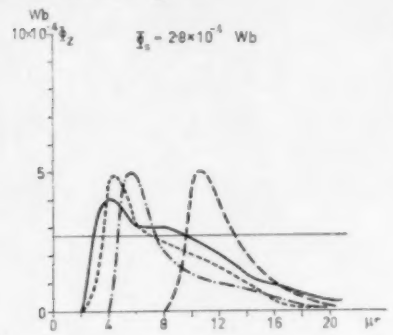

FIG. 10.

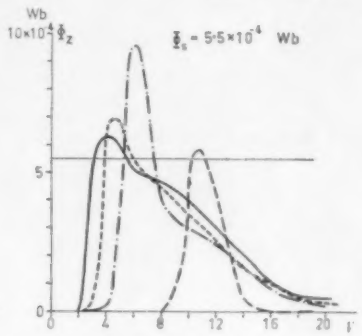

FIG. 11.

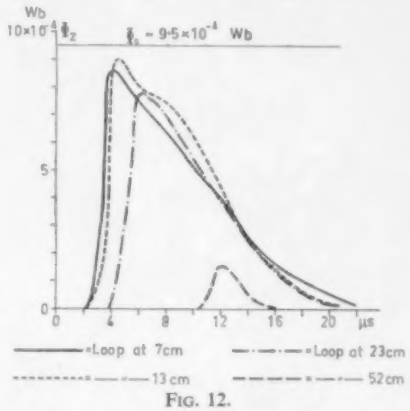

FIG. 12.

FIGS. 10–12.—Flux measured by four loops at different positions for three values of the static radial flux. The curves in each picture originate from four different shots.

When the radial magnetic field is applied, the plasma front becomes much more smooth and the plasma tends to expand towards the walls. Figs. 7(a) and 7(b) are taken with a relatively strong magnetic field (static flux $\phi_s = 5{\cdot}5 \times 10^{-4}$ Wb). This flux value corresponds to a radial field in the middle of the plasma gun of 0·06 Wb/m². The luminous plasma has a shape like a mushroom turned upside down, the current from the central rod seems to continue as a pinch. The plasma expands towards the glass walls and a luminous ring is visible at the line of contact between the plasma front and the wall (see Fig. 7b). When the polarity of the central rod is negative, the central pinch emits much stronger light (Fig. 8).

Due to the high conductivity of the plasma the static magnetic flux is trapped in the hole of the ring and is pushed forward in front of the plasma. It is measured by the loops around the glass tube. An example of the oscillograms is shown in Fig. 9 which gives simultaneous flux oscillograms for two loops situated at $z = 7$ and $z = 17$ cm from the gun.

When the direction of the static magnetic field is reversed, the transported flux has also opposite direction. If the polarity of the discharge is reversed, there is no appreciable change in the oscillograms.

5. INTERPRETATION OF FLUX OSCILLOGRAMS

Diagrams 10–12 show the flux measured by four loops at different positions as a function of time and

This flux is obviously due to a current circulating in the plasma, and cannot be due to the main discharge current, which according to Fig. 3 passes zero at 13 μsec which is not very far from the maximum flux through the last loop. This is a direct indication that a free moving ring has been formed.

When the radial flux is as high as $9{\cdot}5 \times 10^{-4}$ Wb (Fig. 12) the ring is not definitely separated from the gun. The strong field pushes the plasma outwards against the wall, where it has recombined almost totally before it reaches the last loop. Hence the magnetic field measured by this loop is very small according to Fig. 12.

The experiment has been repeated using a smaller capacitance in the condenser bank (120 μF). This results in an earlier passage of the current through zero and a somewhat slower motion of the plasma front, so that the flux measured by the most distant loop is maximum when the discharge current passes through zero (Fig. 13). From the instantaneous flux values measured by the different loops at 7·2 μsec, the flux picture is sketched (Fig. 14a).

A similar picture for 9·4 μs is shown by Fig. 14(b). Outside the glass tube the field must satisfy Laplace's equation, and at great distance it must approach a dipole field. Considering this, it must be concluded that a great deal of the flux lines have closed around the plasma ring. The detailed flux picture inside the glass tube is of course not known. The luminous region, as seen from photographs taken at 7·2 and

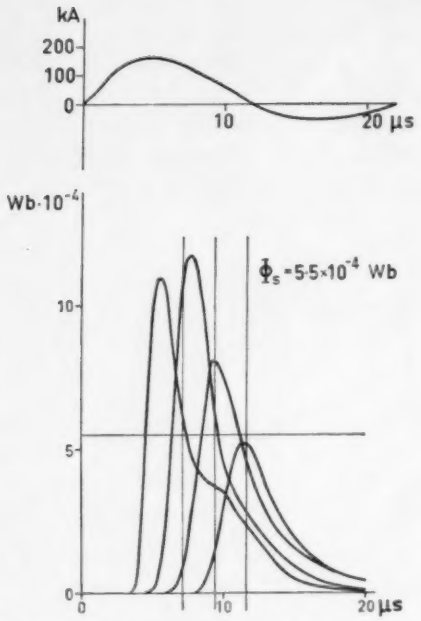

FIG. 13.—Flux measured by four loops at 17, 27, 37 and 47 cm from the gun. At the top a current oscillogram. Capacitor bank reduced to 120 μF. The curves are obtained from two shots, one for the two upper loops, one for the lower ones.

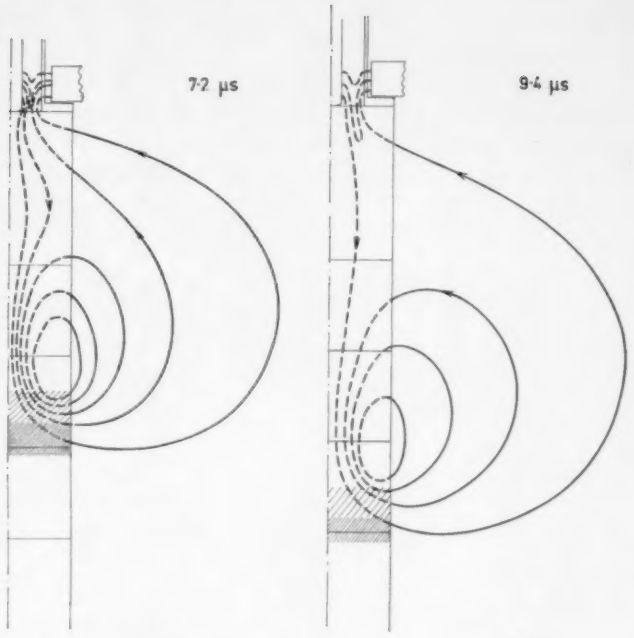

FIG. 14.—Sketches of the field around ring of magnetized plasma based on the instantaneous flux values recorded in Fig. 13.
(a) 7·2 μsec
(b) 9·4 μsec
after ignition.
The hatched areas indicate the luminous regions at the same instants.

9·4 μsec, is marked by the hatched areas in Fig. 14. The maximum flux appears about 10 cm behind the luminous front, and this fact has been confirmed by several measurements.

In each of the diagrams 10–13 the static radial flux is represented by a straight line. Especially for low values of static flux, the flux measured by the loops often has peak values much in excess of the static value. This seems not to be produced by an asymmetry—as might be thought. Instead it seems to be due to a real increase of the flux which is captured by the plasma ring. The peak value of the discharge current has an influence upon the flux transported by the plasma. By altering the number of condensers the discharge current can be varied, whilst the voltage is maintained constant. From a series of such measurements it is found that there is a sharp limit for the discharge current, below which no magnetized plasma leaves the gun. Above this limit, the velocity of the moving plasma front varies very little with the current, e.g. it changes from 5 to 6 cm/μsec when the current is increased from 125 to 250 kA.

In one experiment small rods were placed across the drift tube. When the plasma reached them a pattern of shock waves was produced (Fig. 15). It is believed that a closer study of this phenomenon will be important for the diagnostics of the plasma and also for the study of collisions between a plasma and a solid body.

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Discussions with Dr. N. HERLOFSON, Mr. A. DATTNER, Mr. E. DAHLBERG, Dr. S. LUNDQUIST and Dr. C. T. CHANG have been very stimulating.

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Mr. J. E. LITSENIUS and Mr. C. ZALENSKI of the Research Institute of National Defence have kindly provided the Kerr-cell cameras.

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Note added in proof—In Section 1 it is mentioned that experiments have been started in order to bring the rings to rest. These experiments have been successful and rings with life-times of about 40 μsec have been produced. As in this experiment all the currents in the plasma gun are damped out before 20 μsec from the start, a free and reasonablaly stable plasma ring is produced which is independent of all external currents.

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A LIQUID CONDUCTOR MODEL OF THE HOLLOW PINCH

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Abstract—Electric current is sent through a hollow jet of mercury which is flowing over the surface of a glass tube. The jet is broken up by the action of the electrodynamic force. However, if at the same time a strong current is sent through a central conductor placed inside the glass tube, the jet can be stabilized. This occurs when the currents are parallel. Sausage instabilities have not been observed in the present experiment.

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1. INTRODUCTION

ONE of the basic problems of controlled nuclear fusion is to trap a hot ionized gas in a magnetic field. There are a number of different ways in which a magnetic 'bottle' may be constructed for this purpose. One possibility is to use the field generated by a solid current loop (LEHNERT, 1958, 1959; COLGATE, 1959). A straight configuration which becomes the closest analogy to the current loop consists of a cylindrical shell of ionized gas embedded in the field of a central conductor which is traversed by an electric current. Hollow pinches of this type have already been discussed by BAKER and ANDERSON (1956) and ANDERSON *et al.* (1958a, b).

The stability of a homogeneous mercury jet traversed by electric current has already been investigated by DATTNER *et al.* (1958). The present experiment concerns the behaviour of a hollow mercury jet and may be regarded as a liquid conductor model of the hollow pinch. However, such a model reproduces only part of the phenomena in a high current discharge. It should be kept in mind that the field lines are not 'frozen' in a mercury jet and, therefore, important differences exist between experiments with ionized gases and with mercury.

2. APPARATUS

The experimental arrangement is shown in Fig. 1. A glass tube is centred in a position along the axis of symmetry of a copper funnel and rests with its lower end against a copper block. Inside the tube a copper rod is situated which is in contact with the copper block. Mercury is poured into the funnel and forms a hollow jet which runs along the glass tube. Whirls at the inlet of the funnel are damped by a baffle with radial strips. The mercury jet has a free length of 105 mm, an outer radius of about $R = 2{\cdot}40$ mm and

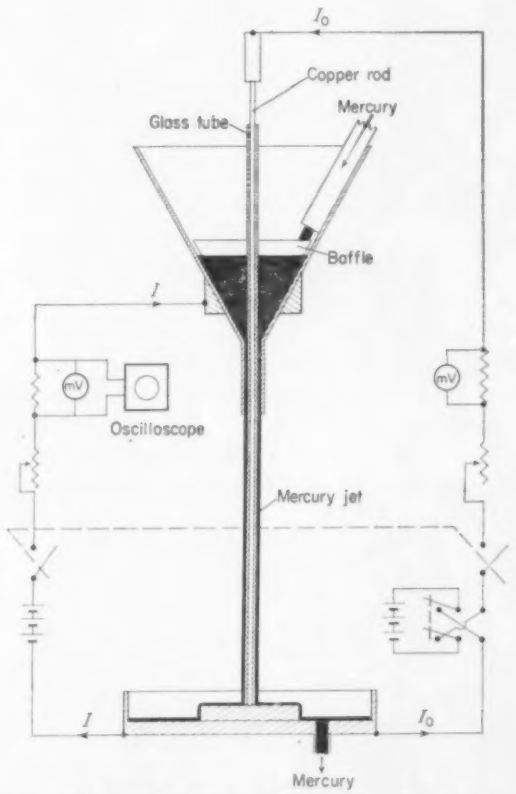

FIG. 1.—The apparatus produces a hollow mercury jet centred around a glass tube. A copper rod is placed inside the tube. Currents may be sent both through the jet and the rod.

an inner radius $R_0 = 1{\cdot}825$ mm. A current of strength I can be sent through the jet. A current of strength I_0 can be sent in arbitrary direction through the central conductor rod. Accumulators are used as current sources and electromagnetically driven switches control the operation of the circuits. An instability of the mercury jet is easily discovered visually as a deformation of its surface, often followed by a break-up and arcing. The instability may also be observed by means

of an oscilloscope connected across the jet or across a resistor in the electrical circuit of the jet as shown in Fig. 1.

3. EXPERIMENTAL RESULTS

When the currents I and I_0 are varied the jet behaves in a way which is illustrated by Fig. 2. Provided that the glass tube is adjusted correctly the jet forms a well-defined cylindrical shell around the tube when no currents are flowing (Fig. 2a). With a sufficiently strong current traversing the jet the latter cannot be kept in its centred position and is displaced sideways. The break-up develops in its earliest stage in the form of a disturbance at the lower end of the jet (Fig. 2b). If a current I_0 is sent through the central conductor at the same time and in a direction parallel with that of I the instability may be delayed. Fig. 2(c) shows a situation where it has nearly been suppressed and only a slight 'kink' remains at the lower end. If the current I_0 is chosen strong enough compared with I the instability may be suppressed completely (Fig. 2d). The jet then resembles a polished steel rod, with even less irregularities in its surface than in absence of any current and there is no sign of any 'sausage' instability. Finally, if the current in the central conductor flows in a direction anti-parallel to that of the jet the latter breaks up violently (Fig. 2e).

A number of measurements has been performed to determine the regions of stability and instability as shown by Fig. 3. A ring indicates the values of the currents I and I_0 for which the jet has been found to be stable and a dot gives the corresponding values for

FIG. 2.—The behaviour of the jet at different currents I and I_0 traversing the jet and the central conductor, respectively. Values are given at the bottom of the Figure.

(a) No currents flowing.
(b) Instability produced by a current I in the jet.
(c) Instability almost suppressed by means of a current I_0 in the central conductor when I_0 is parallel with I. A slight kink is developed at the lowest part of the jet.
(d) The jet stabilized by a sufficiently high current in the central conductor.
(e) Instability is accelerated by sending an anti-parallel current through the central conductor.

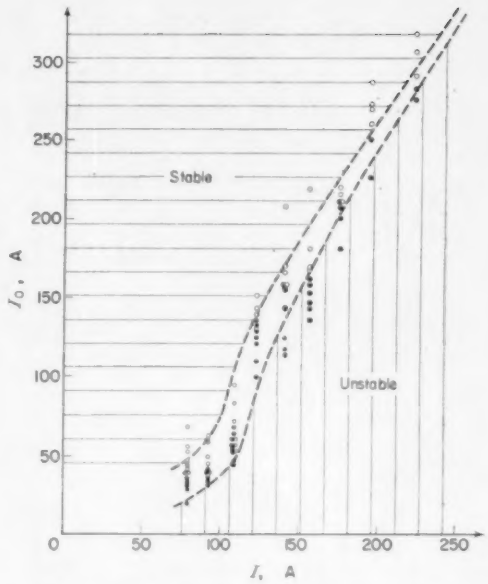

FIG. 3.—Experimental determination of the regions of stability and instability of the hollow mercury jet. Rings give measurements where the jet has been found to be stable and dots where it is unstable. Only points within the transition region are shown in the diagram. In the upper, shaded area to the left the jet is definitely stable; in the lower to the right it is definitely unstable.

the unstable situation. The shaded areas give regions where the jet is definitely stable or unstable, whereas the transition region between the dashed lines represents cases where the conclusions are somewhat uncertain and depend upon errors in the measurements. The transition becomes sharper at strong currents and large electrodynamic forces than at weak currents.

4. DISCUSSION

A rigorous theoretical analysis of the instabilities observed is not within the scope of this paper. Only some elementary considerations will be made on 'sausage' instabilities. According to Fig. 4 the electrodynamic 'pressure' at the inner surface $r = R_0$ becomes

$$f = \int_P^Q iB\,dr \qquad (1)$$

at a trough (P_1) or a crest (P_2) of the 'sausage' disturbance, where i is the current density and B the magnetic field. This gives

$$f = \frac{\mu_0 I}{2\pi^2(R^2 - R_0^2)} \times \left[I_0 \log \frac{R}{R_0} + \frac{1}{2} I - I \frac{R_0^2}{R^2 - R_0^2} \log \frac{R}{R_0} \right] \qquad (2)$$

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and

$$\frac{df}{dR} = -\frac{\mu_0 I}{2\pi^2 R(R^2 - R_0^2)^2} \times \left[2R^2 I_0 \log\frac{R}{R_0} - (R^2 - R_0^2)I_0 + (R^2 + R_0^2)I - \frac{4R^2R_0^2}{R^2 - R_0^2} I \log\frac{R}{R_0}\right]. \quad (3)$$

In the special case where $I_0 = 0$ and the jet is homogeneous ($R_0 \to 0$) the result becomes

$$\frac{df}{dR} = -\frac{\mu_0 I^2}{2\pi^2 R^3}. \quad (4)$$

If, instead, a thin hollow jet with thickness $a = R - R_0$ is considered equation (3) gives

$$\frac{df}{dR} = -\frac{\mu_0 I}{4\pi^2 R^3}(I_0 + 2/3I) \cdot [1 + \mathcal{O}(a/R)], \quad (5)$$

where $\mathcal{O}(a/R)$ indicates terms of order a/R and higher orders.

The result (3) shows that $df/dR < 0$ for a homogeneous jet as well as for a hollow jet when the currents I and I_0 are parallel. Then, the driving forces tend to increase the 'sausage' disturbance and the jet should be unstable. However, for a given area $\pi(R^2 - R_0^2)$ and given currents a comparison between equations (4) and (5) shows that this tendency will be much less developed in a thin, hollow jet with a large radius ($a/R \ll 1$) than for a homogeneous jet. This conclusion possibly may be related to similar results obtained by SMITH (1958) for a cylindrical sheet plasma. Here 'sausage' instabilities are expected to grow more slowly than in a homogeneous pinch. This may also be one of the reasons that such instabilities have not been observed in the present experiment but exist in the experiment with a homogeneous jet (DATTNER *et al.*, 1958). In the latter the largest modulus of df/dR turns out to be about $1{\cdot}3 \times 10^6$ N/m^3, whereas the corresponding value in the former is about $5{\cdot}2 \times 10^5$ N/m^3, i.e. about two and a half times less. However, the action of the surface tension may also affect the results.

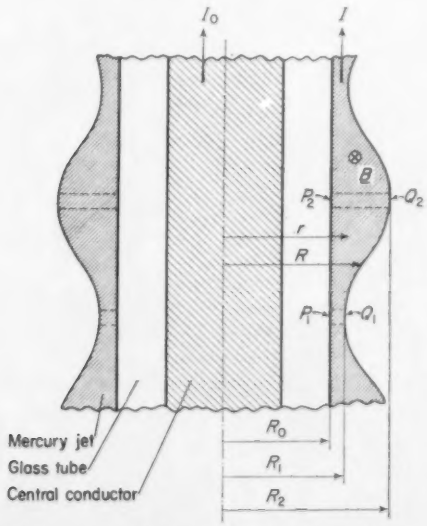

FIG. 4.—Section of the jet along the axis of symmetry. The outer surface has been deformed by a 'sausage' disturbance.

The present experiment has shown that it is possible to stabilize a hollow mercury jet for a sufficiently long time by means of the magnetic field from a central conductor. However, an extrapolation of the results to gas discharge experiments is questionable. It is certain that in a mercury model the current pattern inside the fluid and the corresponding magnetic field will change when the fluid surface is displaced in some way. However, these changes are of a special form and are sometimes very far from the changes obtained in a situation where the field lines are completely 'frozen in'. In particular, magnetic fields generated by sources outside of the fluid volume are not 'frozen in' and will hardly be influenced by the fluid motion.

Other important differences exist between the present experiment and the gas discharge experiments by BAKER and ANDERSON (1958) and ANDERSON *et al.* (1958a, b) on the 'Triax' sheet pinch devices. In the mercury experiment the inner surface of the jet leans against a rigid wall and its stability properties cannot be studied. In the 'Triax' the sheet has two free surfaces. Further, it should be observed that the current in the central conductor of the 'Triax' is antiparallel to the current in the sheet and has a value less than the latter.

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PENDULUM ANALOGUES TO ILLUSTRATE THE PARTICLE MOTION IN SPIRAL RIDGE ACCELERATORS

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Abstract—The essential features of the motion of a particle in an azimuthally varying magnetic field of 'spiral ridge' form may be described by a pair of non-linear coupled equations of simple form. These reduce to the uncoupled linear radial and vertical betatron equations when the azimuthally varying component of field is zero. In this paper two pendulum analogues are described, which obey the same equations of motion as the particles. The first of these represents the radial motion only and consists of a torsional pendulum with a magnetic moment which interacts with a rotating magnetic field; the second, which represents both radial and vertical motion, consists of a compound pendulum supported by two sets of pivots at right angles, with a magnetic bob which interacts with a travelling magnetic field. The first of these analogues was used to make an extensive survey of the maximum stable amplitude of the motion as the parameters corresponding to ridge angle, azimuthal field modulation depth and radial field index were varied. No quantitative measurement was made with the second model, but the effect of ridges in stabilizing the vertical motion was demonstrated.

1. INTRODUCTION

ALTHOUGH at the present time any accelerator incorporating spiral ridge guiding fields (SYMON *et al.*, 1956) would certainly be designed with the aid of a large computing machine, so that a simple analytical form of the field is not necessary, the consideration of a simple field law helps one to understand the basic physical features of the motion. Thus for a field of form (Fig. 1)

$$B/B_0 = (1 + k\rho)[1 + \delta \sin N(\theta - \rho \cot \sigma)] \quad (1)$$

the approximate equations of motion, valid for small ρ and ζ and $\cot \sigma \gg 1 \gg k/N^2$ are (KING and WALKINSHAW, 1958):

$$\rho'' + (1 + k)\rho = -\delta \sin N(\theta - \rho \cot \sigma) \cosh (N\zeta \cot \sigma) \quad (2a)$$

$$\zeta'' - k\zeta = -\delta \cos N(\theta - \rho \cot \sigma) \sinh (N\zeta \cot \sigma) \quad (2b)$$

where the symbols are defined at the end of the paper. For $\delta = 0$, these reduce to the well known betatron equations with $k = -n$; for $\delta \neq 0$ the terms on the r.h.s. modify the motion, and can restore stability in the vertical plane.

Since ζ is small the cosh x and sinh x terms on the r.h.s. can be replaced by unity and x respectively. The r.h.s. in the radial equation therefore represents a harmonic forcing term, the phase of which is proportional to the amplitude, of the motion. In the vertical equation the r.h.s. represents a forcing term proportional to amplitude, and with phase again linearly dependent on the amplitude of the *radial* motion. Such forcing terms lead to an interesting form of non-linear motion, and their simplicity suggests that some analogue might easily be set up. This indeed proves to be the case, and two analogues will be described in succeeding sections. Pendulum analogues of a rather different form have been constructed by BARBIER (1956) for the study of motion in the alternating gradient synchrotron.

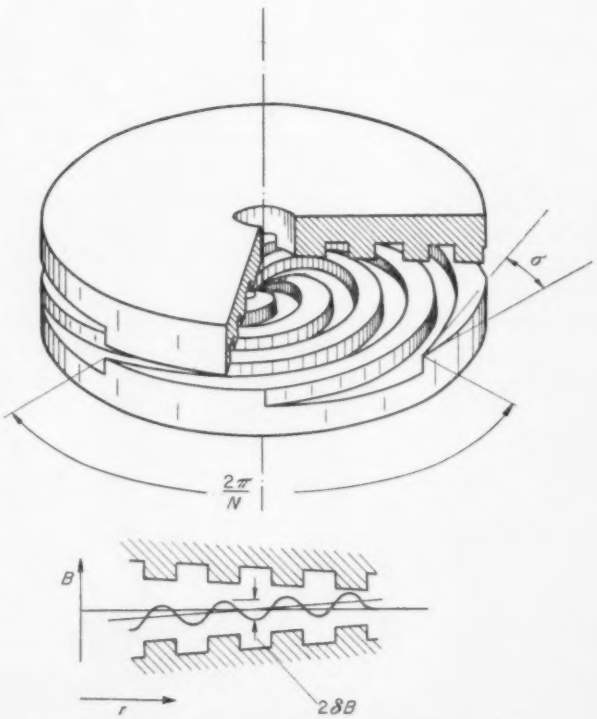

FIG. 1.—Cut-away view of a spiral-ridge pole, and variation of field with radius.

2. ANALOGUE OF RADIAL MOTION

An analogue of the radial motion only may be provided by suspending a magnet to form a torsional pendulum, subject to a rotating field as shown in Fig. 2. The equation of motion of the system is

$$I\ddot{\phi} + S\phi = -MB \sin (p\Omega t - p\phi) \qquad (3)$$

which is of the same form as equation (2a). (If the second term on the l.h.s. were absent, corresponding to the free suspension of the magnet, then the system would be similar to a loss-free synchronous motor running with no load.)

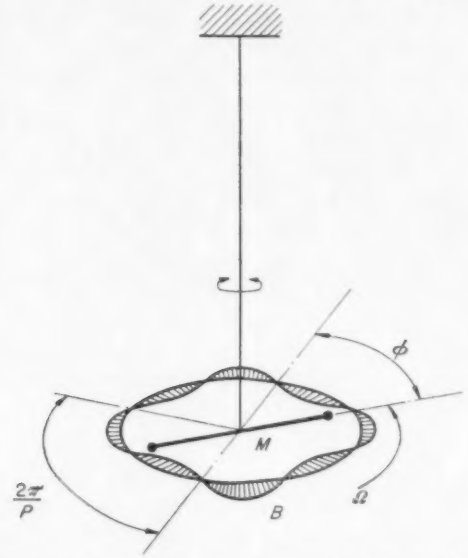

FIG. 2.—Diagrammatic view of suspended magnet M and the surrounding magnetic field, B.

Equations (2a) and (3) may both be written

$$\xi'' + (T_2/T_0)^2\xi = -(T_2/T_1)^2 \sin (\eta - \xi) \qquad (4)$$

where the variables are related as shown in Table 1. The T's represent various characteristic times as shown in Table 2. The reason for this formulation is that the times associated with the pendulum may be measured directly, so that I, S, M and B need not be separately determined.

TABLE 1.—RELATION OF VARIABLES

Equation (4)	Equation (2a) (accelerator)	Equation (3) (pendulum)
η	$N\theta$	$p\Omega t$
ξ	$\rho N \cot \sigma$	$p\phi$
$\left(\frac{T_2}{T_0}\right)^2$	$\frac{1+k}{N^2}$	$\frac{S}{Ip^2\Omega^2}$
$\left(\frac{T_2}{T_1}\right)^2$	$\frac{\delta \cot \sigma}{N}$	$\frac{MB}{Ip^2\Omega^2}$

TABLE 2.—DEFINITIONS OF T

Accelerator	Pendulum
T_0 Free oscillation period in ridge-free field ($\delta = 0$).	Free oscillation period in the absence of rotating field.
T_1 —	Period of small oscillations of freely suspended pendulum in the presence of stationary magnetic field ($\Omega = 0$).
T_2 Time for particle to move through one period of ridge structure, $=2\pi/\omega N$.	Time for magnetic field to rotate by 2π, $=2\pi/p\Omega$.

The values of T appear in Table 1 as ratios. Individual values have the significance shown in Table 2.

A pendulum was constructed having the form shown in Fig. 3. A magnetic multipole consisting of sixteen iron-cored electro-magnets mounted on a central iron ring with alternate polarity was suspended at the centre of a steel wire held under tension. The

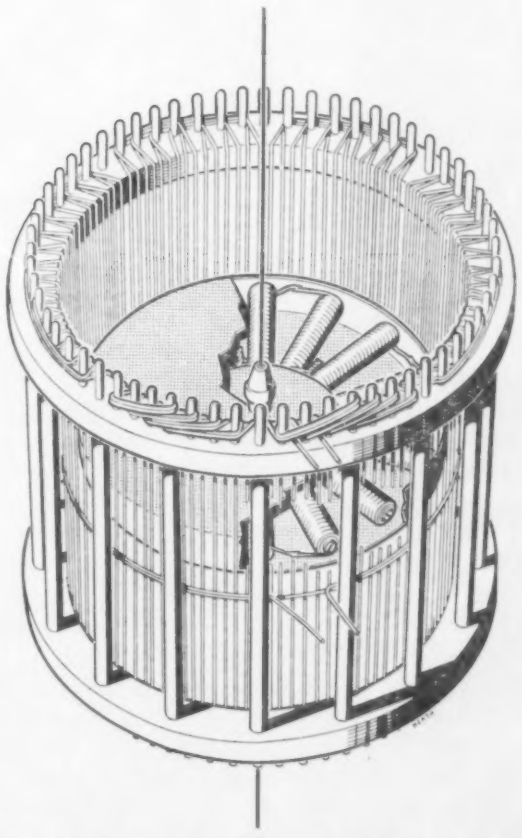

FIG. 3.—Cut-away view of torsional pendulum, showing multipole magnet and wire array.

multipole was surrounded by a smooth cylindrical casing to reduce air-damping. Two helically wound strips of copper foil, mounted in torsional opposition, provided the electrical connexions to the magnets. A fixed cylindrical array of wires surrounded the multipole system. Time varying currents through these wires produced a rotating harmonic wave with eight wavelengths measured circumferentially, ($p = 8$). Harmonics with higher periodicity both in angle and time were present, but their effect on the motion was small. The values of the currents in both the multipole and the wire array could be varied. The diameter of the array was 4 in. and the period of oscillation was about 2 sec.

The characteristics of the motion may be conveniently exhibited on the $\xi - \xi'$ plane at intervals of 2π in η; such a diagram is known as a 'phase-plot' and individual points as 'phase-points'. To obtain such a plot the motion of the pendulum was observed and recorded optically by means of a small mirror of low inertia attached to the wire suspension of the pendulum, together with the appropriate optical system and a light sensitive recording mechanism. A continuous photographic trace was thereby obtained describing the motion of the pendulum, and superimposed upon this trace were placed time markers corresponding to the time interval for the rotating magnetic field pattern to advance by one wavelength. Subsequently, these traces were analysed and converted into phase-plots by means of a simple optico-mechanical device, (RUSSELL, 1958).

To obtain information about the pendulum motion more quickly than could be done by the above method an alternative procedure was later used. The pendulum motion was used to modulate the intensity of a light beam incident upon a photocell, the amplitude and rate of change of amplitude of the photo-electric currents being proportional to ϕ and $\dot{\phi}$ respectively. These two signals were applied simultaneously to the X and Y plates of an oscilloscope, the spot intensity of which was triggered at the periodicity of the rotating magnetic field. The succession of points appearing on the screen develops a phase-plot which can be recorded photographically.

3. DESCRIPTION OF PHASE-PLOTS

At the time of these experiments the general features of the motion were understood, and numerous phase-plots had been computed for equations similar to, but rather more complicated than, equation (4) (LASLETT and SYMON, 1956; KING and WALKINSHAW, 1958). The experiments provided a broad survey of the behaviour of the system for values of $(T_2/T_1)^2$ from 0·1 to 0·3 and $(T_2/T_0)^2$ from 0·01 to 0·1; this is the range of parameters of likely interest in spiral ridge accelerator design.

Although the experiments mainly served to confirm and extend slightly the known properties of the solutions of equation (4), obtained from general arguments and the study of some particular cases, it is worth attempting a general survey of what was found. Let the solution of (4) be $\xi = f(\eta)$, and let the points $\xi = f(\eta_0 + 2\pi n)$, where n is an integer, be plotted on the ξ–ξ' plane. In general it is found that for some particular value of T_2/T_1 and T_2/T_0 in equation (4) several types of solution exist. For one class of solutions the phase points define a closed curve, a series of closed curves or a finite set of points; for a second class no such closed curves or sets of discrete points exist. These classes will be designated 'stable' and 'unstable' respectively, even though in the latter class the amplitude does not increase indefinitely. There is always one and only one solution for which the phase-plot is a single point. This corresponds to an orbit in the accelerator which closes after one turn; such a closed orbit is periodic in η with 2π.

The phase-plot for a solution near to the closed orbit will, in general, be a smooth closed contour encircling the central fixed point. If after m circuits of the contour n points have been plotted, then m/n tends to a finite limit q as m tends to infinity. The parameter q is of order T_2/T_0; it corresponds to Q/N for contours arbitrarily close to the central fixed point, where the system may be shown (by simple expansion of equation (2)) to be linear. Solutions near the closed orbit form a family of similar, smooth, closed contours, the phase points defining any particular contour being approximately uniformly distributed round that contour. The contours degenerate into a set of n fixed points if m/n tends to a rational fraction, the motion then being periodic with $2\pi n$. These fixed points are of special significance if m is equal to unity.

As q approaches $1/n$ the contours in some cases distort into irregular shaped figures with n 'corners' as shown for example in Fig. 4(a) for $n = 5$. Under these circumstances the phase points concentrate near the corners, and when $q = 1/n$ the limiting contour degenerates into n fixed points. Just outside the limiting contour the phase points describe curves similar to those inside the limiting contour in the region between the fixed points; near the fixed points however, successive phase points move outward in a direction away from the central fixed point. This outward movement extends to a distance of order 2π in ξ or ξ', thereafter the phase points occur in a seemingly

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Fig. 4(a).

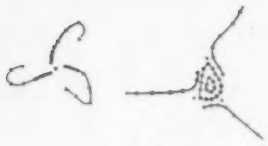

Fig. 4(b).

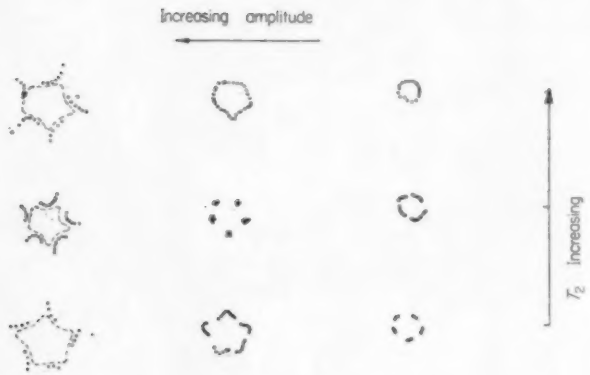

Fig. 4(c).

Fig. 4(d).

FIG. 4.—(a) Phase-plot showing both stable and unstable motion, $q \approx 1/5$.
(b) Typical plots near the $q = 1/3$ resonance.
(c) Sequence of plots showing the existence of a finite stable area when $q = 1/5$.
(d) Plot showing island formation with $q = 2/9$. The contour just inside the islands is in a region where the shape of the plot is a critical function of amplitude, and its erratic shape may be due partly to damping and other experimental effects.
(Plots *a* and *d* were made by the first method described, plots *b* and *c* were made by the second.)

erratic manner. In this third region it is difficult to join the points up in a meaningful manner, since lines joining the points would have to cross. This is not allowed, since specifying ξ and ξ' determines the subsequent motion uniquely. These characteristics correspond to unstable motion.

The type of transition from stable to unstable motion described above always occurs when q is equal to one of the values 1/2, 1/3, 1/4, and sometimes occurs for integral values of $1/q$ greater than 4. As the q for arbitrarily small amplitudes (which is a function of T_2/T_0 and T_2/T_1) approaches one of the values 1/2, 1/3, or 1/4, the area enclosed by the limiting contour for stable motion tends to zero. This is shown for the $q = 1/3$ sub-resonance in Fig. 4(b). For q equal to 1/5, 1/6, 1/7 or 1/8 finite areas were found to exist, although considerable reductions occurred in the maximum areas enclosed by the limiting stable motion contour. A sequence of diagrams illustrating the finite area for $q = 1/5$ is shown in Fig. 4(c).

In the region between the main resonances the normally closed contours may under certain conditions break into a set of smaller closed contours, termed 'islands'. The number of islands is of the form $An + B(n + 1)$ where A and B are integers and n, $n + 1$ are the $1/q$ values of the adjacent main resonances. The formation of islands may be interpreted as the motion 'locking in' to the structure. An example of island formation with $q = 2/9$ is shown in Fig. 4(d). The shape of the closed contour just inside the islands is very dependent on the initial conditions, and the very erratic shape may be to some extent due to imperfections in the pendulum. Islands did not always appear to be present between the main resonances, and for q values given by $m = 2$ and n equal to 5, 7 or 9, the diagrams sometimes showed instability for n fixed points as for the main resonances.

Fig. 5 shows the variation of stable phase area with T_2/T_1 and T_2/T_0; the dips corresponding to resonances can clearly be seen. It was often difficult to define the stable limit accurately (especially when islands were present) and consequently the figure should be taken as qualitative only. The precise value of the limit is of academic interest only, since when vertical motion is taken into account the stable limit will be smaller.

4. ANALOGUE OF RADIAL AND VERTICAL MOTION

It is possible to design a pendulum with a magnetic 'bob' moving in a simple coil system, which obeys equations of motion analogous to equations (2a) and (2b). A compound pendulum with two independent supports at right angles, one of which is above the

centre of gravity and one below will move in a manner described by equation (2) with the r.h.s. equal to zero. The forcing term on the r.h.s. may be provided by means of a coil array as shown in Fig. 6. If the first harmonic of the wave only is considered, then it is easy to show that the forces exerted on the bob are of the form given by the r.h.s. of equation (2). The equations of motion of the pendulum in terms of the fundamental parameters of the system (moment of inertia, length, etc.) are complicated, and as with the radial analogue, the system is best described in terms of characteristic times, thus

$$\left.\begin{aligned}\xi'' + (T_2/T_0)^2\xi &= -(T_2/T_1)^2 \sin(\eta - \xi)\cosh\psi\\ \psi'' + (T_3/T_0)^2\psi &= -(T_2/T_1)^2 \cos(\eta - \xi)\sinh\psi\end{aligned}\right\} \quad (5)$$

It has been assumed that the moment of inertia about either set of pivots is the same, and this will be very nearly true. The times T are defined as in Table 2 except that in the definition of T_1 'freely suspended' is replaced by 'in the absence of gravity'. Neither T_1 nor T_3 (which is the imaginary period in the 'vertical' plane in the absence of the magnetic field) can be measured directly. However they can be deduced quite simply from other easily measured characteristic times of the system (LORD, 1958).

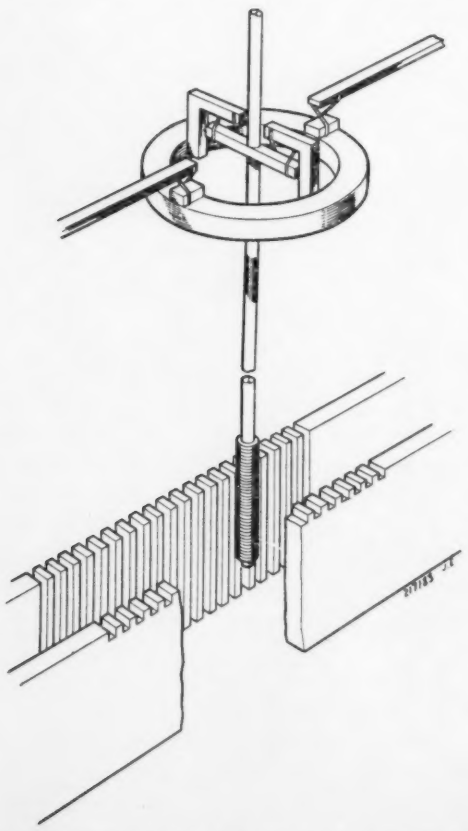

FIG. 6.—Double pendulum system. Wires were wound in the slots shown.

A pendulum of this form was constructed, and the details are shown in Fig. 6. The pendulum consisted of a $\frac{1}{4}$ in. rod, 1 m long, supported by two sets of mutually perpendicular pivots close to its centre of mass. The lower 30 cm of this rod was made of steel; this was magnetized with a coil to form a magnetic pole which interacted with the magnetic driving field. Each pivot consisted of two brass bars one above the other, the lower being suspended from the top one by two pieces of thin piano wire, which crossed over from one side surface to the other. With this pivot system the pendulum had a mechanical Q of about 250 when T_0 was 7 sec.

The array of wires producing the driving field was wound on two parallel plates made of an insulating plastic. The wires passed down $\frac{1}{8}$in. wide slots in the surface of the plates. There were 36 slots, 12 corresponding to one wavelength (3 in.). The currents in the array of wires were switched by a rotary switch through sets of resistors to give a variation of current which was approximately sinusiodal with respect both to time and displacement.

Because of the large number of variables, and the difficulties of setting up the pendulum, no systematic survey was made. Nevertheless the existence of stable and unstable regions, and the method of build-up at resonance could be clearly seen.

LIST OF SYMBOLS

B	magnetic field
I	moment of inertia of pendulum
k	field index defined by equation (1)
M	magnetic moment of pendulum
m	number of circuits of contour in phase plot after n points have been plotted
n	see definition of m
N	number of ridges in field
p	angular periodicity of magnetic field interacting with pendulum
Q	number of betatron oscillations per revolution of the particle
q	limit of m/n as m tends to infinity
S	torque to produce unit angular deflexion of pendulum
T	characteristic time, see Table 2
t	time co-ordinate
δ	angular field modulation in accelerator magnet (equation (1))
ζ	vertical displacement of particle from median plane normalized to the mean radius
η, ξ, ψ	generalized variables, see Table 1 and equation (5)

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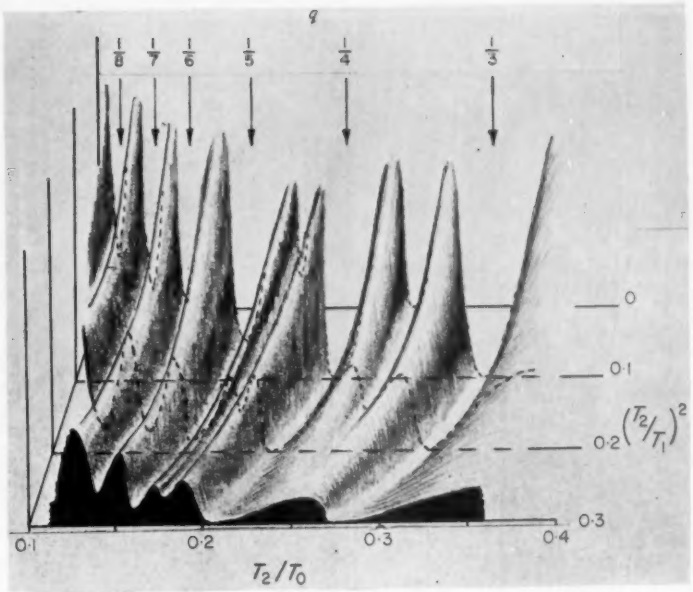

FIG. 5.—Graph showing stable phase space area as a function of T_2/T_0 and $(T_2/T_1)^2$.

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θ angular variable in accelerator
ρ radial variable, measured from and normalized to, the mean radius of the particle
σ angle between spiral ridge and circumference
ϕ angular variable in pendulum
Ω angular vecloity of magnetic rotating wave
ω angular velocity of particle in accelerator.

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THE INJECTOR FOR THE HARWELL PROTON LINEAR ACCELERATOR

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Abstract—The injector is a d.c. accelerator delivering a pulsed proton beam of 15 mA at an energy of 515 keV. The installation is described and details are given of its most important features.

1. INTRODUCTION

THE injection equipment provides a proton beam of 15 mA at an energy of 515 keV. It is basically a conventional d.c. accelerator of the 'open' type, but special problems were encountered in producing a beam of the required characteristics.

It was considered (KING 1954) that the injection equipment would be satisfactory if it offered to the linear accelerator system a proton beam of 10 mA having a radius of 0·25 cm, an angular divergence not exceeding $\pm 0{\cdot}5^\circ$ and with a maximum energy spread of $\pm 0{\cdot}1$ per cent.* The beam was also to be pulsed at a rate of 50 sec^{-1} with a pulse length of 200 μsec. The way in which the equipment was designed and constructed to meet these conditions is described very briefly in this paper.

Protons are extracted from a radio frequency ion source and are focused by a wide-aperture lens system consisting of several electrodes at alternately high and low potentials; the proton beam is then accelerated along a multi-electrode tube containing a uniform electric field. The ion source and the focusing assembly are contained in a corona shield at one end of the accelerator tube, as shown in the foreground of Fig. 1. Various power supplies for both the ion source and focusing electrodes are located in the insulated cubicle to the right. In the background can be seen the cascade generator and the storage capacitors which provide the main high tension. Fig. 2 shows schematically the main components of the installation and the manner in which they are associated. A system of interlocks is used which gives protection to both personnel and apparatus.

We now give a brief description of the various parts of the equipment, with more detailed explanations where these are considered necessary.

* This restriction on energy spread is the result of using a pre-bunching cavity with the linear accelerator.

2. ION SOURCE

The ion source is basically the same as previously reported (THONEMANN and HARRISON, 1955), and uses the principle of coaxial aperture disc electrodes (HARRISON, 1958) for extracting the protons from the radio-frequency excited discharge, as shown in Fig. 3. The ion source has now given service for 3 years and has proved reliable and reasonably trouble-free.

The discharge, running continuously, is excited inductively by a resonant circuit at a frequency of 20 megacycles sec^{-1}. Purified hydrogen is fed into the Pyrex discharge envelope; the pressure is adjusted in the range of 5–10 microns of mercury by a nickel diffusion leak (HARRISON and HOBBIS, 1955 and 1956). Nickel diffusion leaks of this particular kind have now been in use for some time, and for the purpose of supplying a regulated flow of purified hydrogen to ion sources are superior to palladium diffusion devices. A coaxial magnetic field, with a maximum value of approximately 100 oersteds, extends the plasma to the first aperture of the extraction electrodes. The first electrode has an aperture of diameter 0·64 cm (0·25 in.) and is spaced 0·32 cm (0·125 in.) from the second electrode in which there is a short canal of 0·32 cm diameter and 0·32 cm long. The extraction potential is applied between these two electrodes and is pulsed at the rate and for the time interval for which the beam is required. A beam of 20 mA, mostly of hydrogen ions, is readily obtained with a potential difference of 9 kV; this is equivalent to a perveance ($I/V^{3/2}$) of at least 10^{-6}. If desired, the extraction geometry can be modified to give an increased perveance. The proton percentage in the beam normally exceeds 70 per cent.

3. BEAM FOCUSING

The beam is focused electrostatically by a lens system, shown in Fig. 4, consisting of four electrodes

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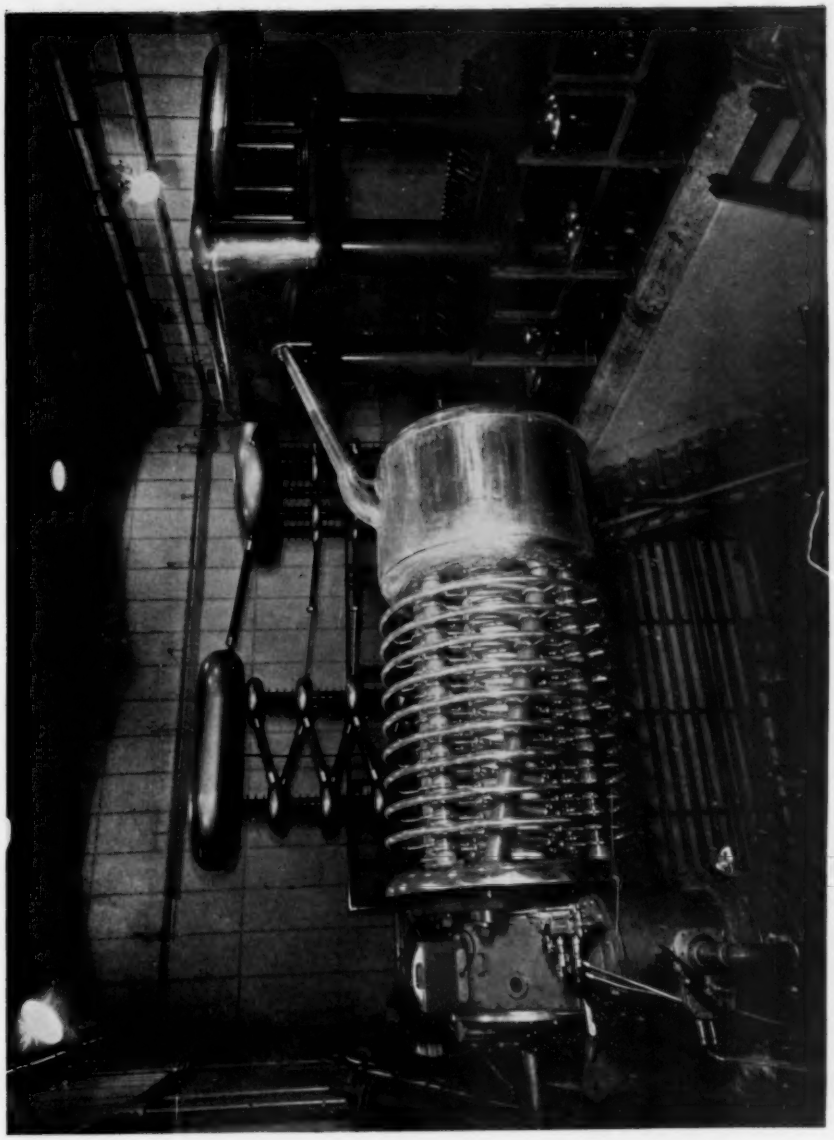

FIG. 1.—General view of injector installation.

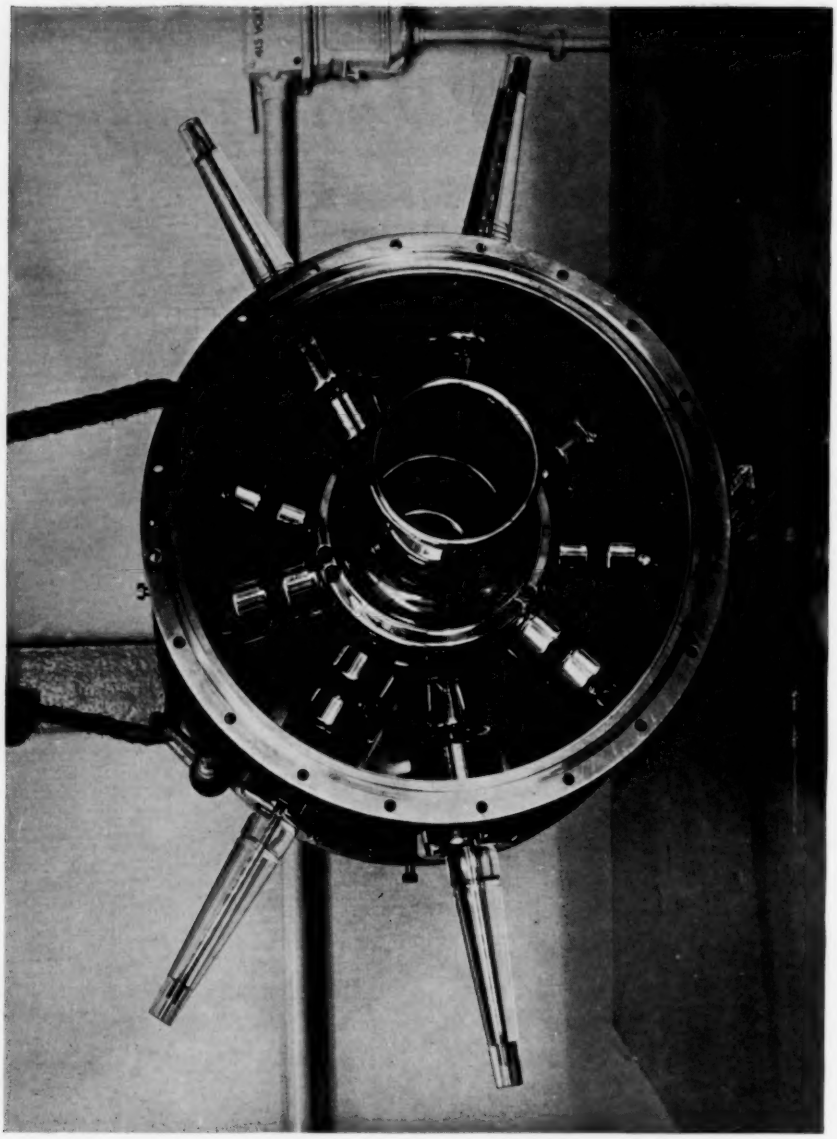

FIG. 5.—Assembled focusing system.

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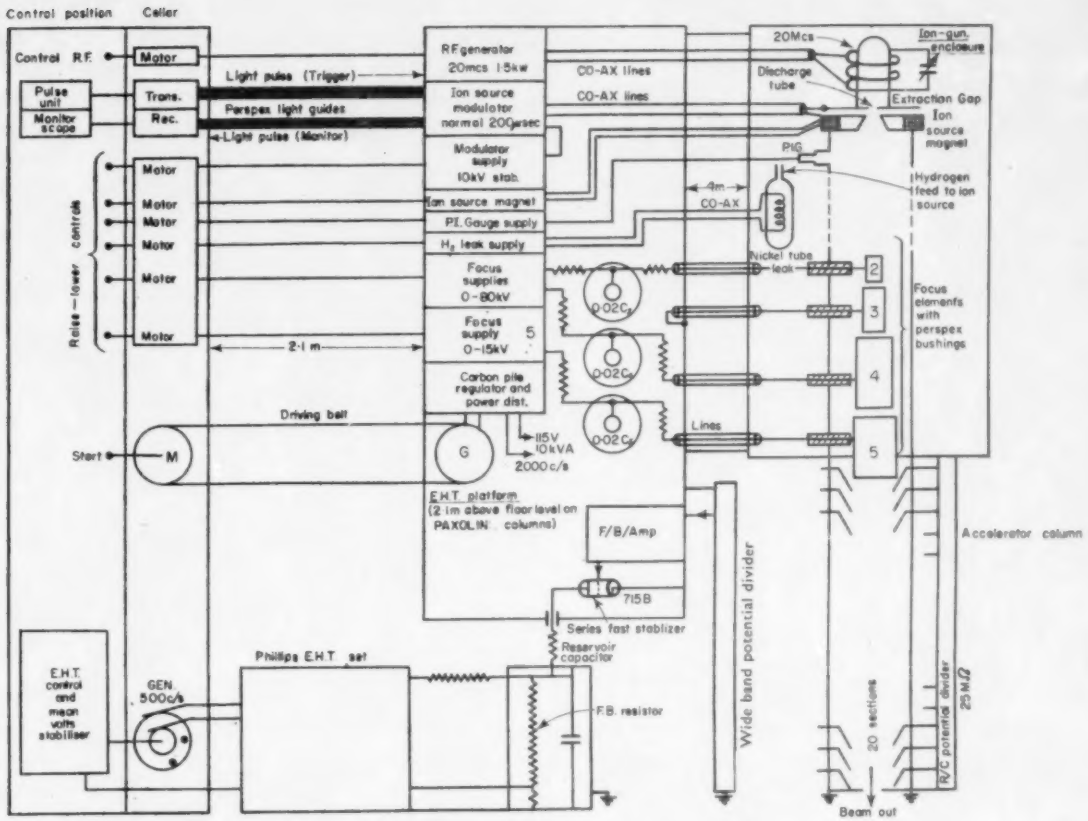

FIG. 2. Schematic arrangement of system.

of progressively increasing diameter. The potentials of the electrodes are controlled independently. Typical operating values, expressed relative to the region of zero proton energy, are

Extraction potential: $V_1 = 9$ kV

Focus electrode potential: $V_2 = 55$ kV

$V_3 = 9$ kV

$V_4 = 45$ kV

$V_5 = 15$ kV

The final design of the focusing system was reached in the following way. Trajectories throughout the accelerating tube and the drift space (the non-accelerating region between the injection equipment and the first radio frequency gap of the linear accelerator) were computed (KING, HOBBIS and HARRISON, 1955) for proton beams adjusted to have a waist in the plane of the first gap of the linear accelerator, where ideally all rays would be parallel. A range of parameters was used and space charge forces were included. For example, in the case of a beam of 20 mA, an electric field of 4 kV cm^{-1} in the accelerator tube, and a drift space of 100 cm in length, it was shown that at an energy of 100 keV the beam should have a radius of approximately 2 cm and be converging at an angle of 2°. At an energy of 40 keV the corresponding values are 3·3 cm and 3°, respectively. It was found that a focusing system consisting of a single gap could not, owing to the effect of space charge, match the beam from the ion source to the trajectories required in the accelerator tube. The possibility was then examined of using several gaps consisting of electrodes of alternately high and low potentials. An approximate theoretical method was used and the results checked by several experimental arrangements. This work confirmed the suitability of the method and led to the design shown in Fig. 4. In one experiment, the space charge dynamic sealing laws (HARRISON, 1957) were used and a $\frac{1}{3}$th scale model of the ion source, focusing system, accelerator tube, and drift space was constructed and its performance studied. Various focusing systems were tested in the model, and the results from this early work were subsequently found to be consistent with the performance of the completed injection equipment.

The maximum focusing potential expected was

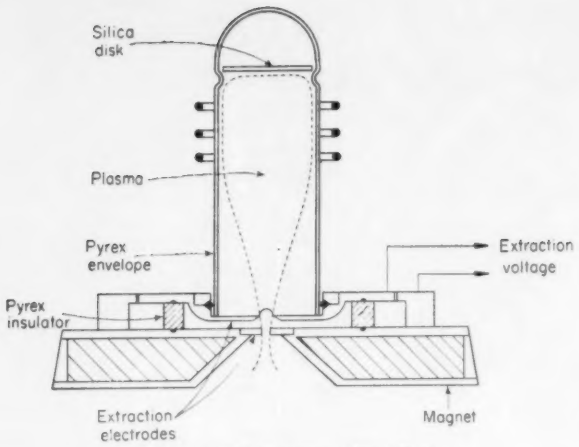

FIG. 3. The ion source (simplified).

80 kV. Supporting insulators for the electrodes, and the insulated bushings for connexion to the electrodes, were developed and tested at potential differences greater than 100 kV; the supporting insulators were made from Pyrex with sand blasted surfaces, and the bushings were made from Perspex. The assembled focusing system, with its stainless steel electrodes, is shown in Fig. 5.

4. ACCELERATOR TUBE

The accelerator tube consists of 20 sections, with a potential difference of 25 kV per section. The sections are porcelain rings of 10 in. internal diameter, separated by steel diaphragms to which the electrodes are clamped; the whole assembly is sealed together with cold-setting Aradlite cement. The electrodes are aluminium, and shaped as in Fig. 4 to protect the insulators from bombardment and screen the beam

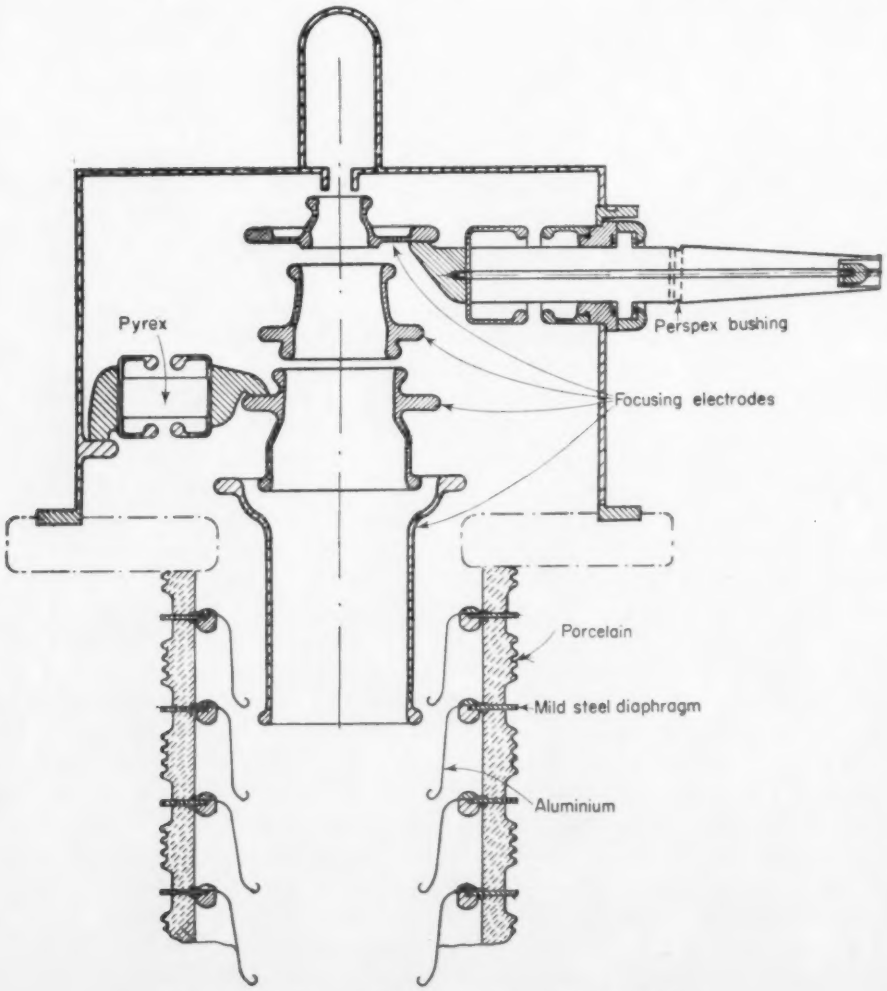

FIG. 4. Focusing electrodes and initial stages of accelerator tube.

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from the influence of charge distributed on them. Up to this stage in the description the accelerator tube is in many respects identical with that described by CHICK and MIRANDA (1952) and CHICK and PETRIE (1955).

The potentials for the electrodes are tapped off from a chain of 25 M ohm resistors supported by corona-free attachments built round the accelerator tube. Provision was made for adding either a second resistor chain or a series of 700 $\mu\mu$F capacitors. It was thought that the electrodes might change their potentials due to beam loading during the period of the pulse, and the capacitors were intended to prevent this effect; so far, however, they have not been necessary. The accelerator tube, including the potential divider and the 1·25 m diameter corona hoops, is supported by a cantilever structure containing four horizontal and two diagonal tie rods made from fibre glass bonded with polyester resin.

The accelerator tube has proved to be vacuum tight; external electrical breakdown has occurred, however, from time to time as a result of corona discharges from the helical tracks of the potential divider resistors. Resistors have also failed by becoming open circuited, possibly because of the corona discharge and electrical breakdown. These difficulties are now avoided by using resistors of the same type but sheathed in close fitting PVC tubing.

5. E.H.T. SUPPLY

The main high tension is supplied from a Philips cascade generator, nominally rated at 700 kV, 8 mA. Smoothing is provided by a 3·5 M ohm resistor and a 0·01 μF storage capacitor which, in addition, limits the potential variations during the beam pulse. The high tension is stabilized at this point to $\pm$0·25 per cent against slow variations in time by feedback control of the excitation of the rotary generator which supplies the input to the high tension equipment; the error signal is determined by reference to a 300 M ohm resistor chain which is used also for measuring the high-tension voltage.

The storage capacitor is connected to the injector equipment through a 10 k ohm damping resistor and a series triode valve. This valve acts as a series stabilizer and forms the last element in a fast response loop. A short-term voltage stability of better than 0·05 per cent is achieved, including the period of the pulse. The long-term stability is not known, but variations occurring over periods of several hours or longer can easily be compensated normally. The loop uses a wide band potential divider, has a response time of 20 μsec and successfully handles total pulse currents of 50 mA.

6. VACUUM SYSTEM AND MECHANICAL DETAILS

The accelerator tube assembly is supported horizontally from its vacuum manifold, which in turn is mounted on a mobile stand which can be moved on rails parallel to the machine axis. The beam is aligned relative to the linear accelerator by movement of the manifold with coarse preset adjustments, and with fine limited motions from remotely controlled motor driven mechanisms.

Two 14 in. diameter oil diffusion pumps, each fitted with a refrigerated baffle at a temperature of -10 to -20°C, are supported from either side of the manifold. Each pump has an unbaffled speed for hydrogen of 3500 l. sec^{-1}, and the minimum pressure recorded with an untrapped ionization gauge is 4×10^{-6} mm of mercury; with the ion source operating the pressure never exceeds 8×10^{-6} mm of mercury.

7. ELECTRONICS

The various supplies and the associated electronic equipment are accommodated in a 3 m $\times$ 3 m $\times$ 1 m cubicle, supported 2·1 m above the floor by 4 paxolin columns, each 0·46 m in diameter. A 2000 cycle sec^{-1}, 10-kVA alternator is driven from a motor situated in a room below floor level by a belt in one of the columns, and supplies the electronic equipment at high-tension level. The remaining columns contain linkages which operate switches, variacs and potentiometers. Air cooling is also supplied for the ion source and the electronic equipment, and in addition carbon tetrachloride is circulated from ground level for direct cooling of the ion source magnet.

Radio frequency power for the ion source is generated by a separately excited 20 megacycle sec^{-1} power amplifier having a variable output, with a maximum value of 1·5 kilowatts. The output is fed through two coaxial cables suitably tapped to the resonant circuit which excites the discharge.

Ions are extracted from the discharge by applying to the ion source electrodes a square voltage pulse, of variable amplitude, from a supply having an impedance of 10 k ohm. A maximum of 9 kV is available, and both pulse length and repetition rate are variable although they are normally fixed at about 200 μ sec and 50 pulses sec^{-1}, respectively. Each pulse is actuated by a light signal, produced by a fast response cathode-ray tube and transmitted by a Perspex light guide to a photomultiplier in the high-tension cubicle. A similar link working in the reverse sense is used for

monitoring at the control desk the extraction potential or current pulse.

The four focusing electrodes have independent variable supplies, although electrode 3 is normally connected to the local earth potential (that is, the high tension). Electrodes 2 and 4 are supplied from small 16 stage radio frequency Cockcroft-Walton generators (WHITBY, 1956) which are oil immersed and driven by 65 kilocycle sec^{-1} valve oscillators. These supplies produce 80 kV at 0·5 mA, they charge 0·02 μF capacitors, and under normal running conditions the output is stable to better than 1 per cent in the range 20–80 kV. Electrode 5 is supplied from a small voltage doubler with a maximum output of 14 kV at 1 mA. Connexion to the focusing electrodes is made through resistors and polythene cables which terminate in polythene mouldings that fit closely over the Perspex bushings shown in Fig. 5.

A direct current stabilized supply of variable output is used for the ion source magnet; there are supplies for the nickel diffusion leak and for the apparatus which monitors the pressure in the high-tension end of the accelerator.

8. PERFORMANCE

The injection equipment described above was completed in 1956. Since its completion the apparatus has worked satisfactorily and a number of tests have shown that it should fulfil closely the anticipated conditions.

Total beam currents of 20 mA, measured calorimetrically (HARRISON, 1957), have been passed through a 1·5 cm diameter aperture placed 1 m beyond the end of the accelerator tube, and in excess of 10 mA has been passed through a 0·5 cm diameter aperture in the same position. Consideration of the optics shows that the current passing through the smaller aperture should lie within the tolerance specified for the angular divergence. Greater current densities could be obtained by using quadrupole focusing magnets between the injector and the linear accelerator. Proton percentages exceeding 70 per cent have been measured without paying any particular attention to the ion source conditions. The protons represented over 80 per cent of the hydrogen ion components; heavier ions up to about mass 30 were observed.

Electron currents in the accelerator tube and X-radiation were minimized by a suppressor electrode system, similar to that described by INALL (1951), installed in the vacuum manifold. X-ray intensities near the injector were considerably above permissible levels and the measurements described above were performed behind steel plates $\frac{1}{2}$ in. thick.

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BOOK REVIEWS

An Introduction to Thermonuclear Research A. SIMON. Pergamon Press, 1959, 182 pp., 35s.

THIS is a most welcome book. Based on a series of lectures given by the author at Oak Ridge in 1955, it gives an accurate and interesting account for the non-specialist physicist and engineer, of a rapidly developing branch of research. It is a useful text also for those actively engaged in this work, who might occasionally wish to turn from immediate problems and review the basic concepts, advantages and limitations of the various approaches to a fusion reactor. At the time these lectures were given thermonuclear work was secret; it is understandable therefore that only American experiments and theoretical contributions are described. However, the book is concerned mainly with fundamental processes, and since British and Russian experiments were at that time developing on similar lines, little of importance—apart from some recent work—has been left out.

In the first chapters Dr. Simon, after describing several fusion reactions between light elements (D, T and He) discusses the energy sources available on the earth from such reactions and the economics of a fusion reactor. The basic conditions to be satisfied are derived, namely a reaction rate high enough to exceed losses and give a net energy gain. This requires that a volume of very hot, and therefore completely ionized, gas be confined well away from material walls, for a sufficiently long time. The most obvious method of confinement is by a magnetic field, and it is shown that losses of particles and energy by motion across a uniform magnetic field can be made satisfactorily small. Motion along field lines is much more serious, and poses the problem of what to do with the ends. Two solutions are discussed, the closing of the lines in toroidal systems (and the consequent drifts due to non uniformity of the field), and magnetic mirrors.

In subsequent chapters the most promising experimental approaches to fusion reactors are described in sufficient detail to show the essential heating processes and problems of confinement connected with them—for example the reduction of outward drifts in the Stellarator by twisting a torus into a figure of eight, methods of injecting ions into magnetic mirrors, and the gross instabilities (wriggling and necking) of the pinch discharge. The author keeps the main practical objective well in view by considering in turn the economic advantages of reactors based upon the Stellarator, magnetic mirror and the pinch effect. The book concludes with an outline of early theoretical work on hydromagnetic stability, and a chapter on diffusion of plasma across a magnetic field, with special reference to ambipolar diffusion and the possible short circuiting of this process when electrons have an alternative path.

The presentation is clear and simple throughout, and developed from concepts of charged particle collisions familiar in classical discharges. Unfortunately, by largely preserving the original lecture material (of 1955), the author has omitted several points the importance of which has since been recognized. Plasma oscillations and electrostatic instabilities, for example, not only destroy confinement of a plasma, but also cause one to re-examine the mixing of particle beams as a method of obtaining an energetic plasma. Cyclotron radiation from electrons promises to be a much more serious source of energy loss than Bremsstrahlung. The idea of 'burn-out' in mirror machines might also have been included with advantage. These are, however, but minor criticisms; it is only a pity that this book was not available several years ago—life would have been much easier for some of us! Now that it is here it is to be most warmly recommended.

G. F.

Plasma Physics and the Problem of Controlled Thermonuclear Reactions Vol. 3. Edited by M. A. LEONTOVICH, translated by J. B. SYKES. Pergamon Press, London, 1959, 422 pp., £8.

THIS is one of the four volumes produced by the Russian delegation to the Second International Conference on the Peaceful Uses of Atomic Energy, as a summary of their work in the controlled thermonuclear field. This volume is largely devoted to applied theoretical works, although there is some pure theory, and six of the twenty-six papers contain experimental results. Many papers describe possible confining configurations. Budker gives a survey of magnetic mirror containment, with an analysis of losses, considered as a steady state diffusion problem with a rather arbitrary source, and a brief and rather optimistic discussion of the direct conversion of thermonuclear energy into electricity. Osovets describes an ingenious device in which a ring discharge is produced and confined in a disk-shaped vessel by an increasing magnetic field; the field varying in space in such a way that the current suddenly collapses to produce a sphere of hot gas. A large number of magnetic probe measurements are reported, on this device, which combines many features of the so called Z and theta-pinches. Two further containment configurations are described by Kadomtsev. In the first of these, separated windings produce a magnetic field with ripples near the surface of the confining vessel, so that field lines are longest near the walls. This, the author claims, prevents the drifts characteristic of toroidal systems, and leads to a stellerator-like configuration, in which the plasma is contained by externally produced magnetic fields. In the second, plasma stability is produced by placing solid conductors as guard rings on the plasma surface which then is concave outwards. A good case is made for this, but no detailed examination is made of the losses at critical cusp-like points near the solid conductors.

One paper on experimental technique describes the intriguing piezoelectric gauge used to measure the transient pressure at the centre of a fast pinched discharge. Among the purely theoretical papers is one by Belyaev, which uses Bogolubov's approximate representation of the correlation between particles in an attempt to derive a rather better fundamental kinetic equation for the plasma than is given by the Boltzmann equation. Another by Sagdeev and Rudakov uses the fact that the Larmor radius is small to derive magnetohydrodynamics from the collisionless Boltzmann equation. Although their work is incomplete since only double adiabatic magnetic-hydrodynamics is derived, they discuss the instabilities arising from asymmetries in the velocity distribution. Finally, a most important paper contains Trubnikov's calculation of the synchrotron radiation from the

electrons in a magnetized plasma. In this he shows that at the high temperatures required for the controlled release of thermonuclear energy, relativistic effects will permit an extremely serious fraction of this radiation to escape from the plasma.

The material consists entirely of research papers, and while of great importance to workers in the field, it might present difficulties to the novice. These difficulties are not diminished by the occurrence of a number of misprints in the mathematics, many of which are also present in the Russian original. The translation is clear and idiomatic, while the book itself, in spite of its price, is reproduced directly from typescript, and thus is adequate rather than elegant.

W. B. T.

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IMPORTANT EDITORIAL NOTICE

In view of the fact that not all the articles appearing in the *Soviet Journal of Atomic Energy* are of interest to our readers, the Board of Editors have decided to publish only such Russian papers which they consider to be within the scope of the *Journal of Nuclear Energy*. However, arrangements have been made, with the kind assistance of the Pergamon Institute (a division of Pergamon International Corporation) and the Academy of Sciences of the U.S.S.R., for the *Journal of Nuclear Energy* to carry translated contents lists and abstracts of the complete forthcoming issues of the *Soviet Journal of Atomic Energy*. A translation of any article which has not been chosen as suitable for inclusion in the *Journal of Nuclear Energy* may be obtained on request from the Pergamon Institute, under their co-operative cost-sharing plan.

All enquiries and orders for translations of Russian articles should be addressed to the Administrative Secretary of the Pergamon Institute at Headington Hill Hall, Headington, Oxon.

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THE SUPPRESSION OF BETATRON OSCILLATIONS IN STRONG-FOCUSING ELECTRON SYNCHROTRONS*

A. A. KOLOMENSKII and A. N. LEBEDEV

(*Received* 28 *October* 1957)

Abstract—Various methods of producing simultaneous radiation damping of synchrotron and betatron oscillations in a strong focusing accelerator are examined and formulae for calculating the damping decrement are given. It is noted that the curvature of the orbit in the damping magnets must be different from that in the main sectors. It is shown that the total damping decrement is always the same and is independent of the type of damping system used.

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1. INTRODUCTION

IN previous reports[1] we have shown that the radiation from the electrons in cyclic accelerators leads to damping or a build-up of the betatron and synchrotron oscillations. In an accelerator having a magnetic field which is axially symmetric, the corresponding decrements are given by the equations

$$\xi_{\rho\,\mathrm{bet}} = -\frac{n}{2(1-n)}\,\Gamma, \qquad \xi_z = -\Gamma/2 \tag{1}$$

$$\xi_{\mathrm{synchr}} = -\frac{3-4n}{2(1-n)}\,\Gamma, \qquad \Gamma = W/E \tag{2}$$

where W is the intensity of the radiation from an electron having energy E, n is the exponent characterizing the magnetic field gradient and the decrements ξ enter into the expression $a \sim \exp(\int \xi\, dt)$ describing the change in the amplitude a of the oscillations.

For synchrotron oscillations in a strong focusing accelerator we obtained the result

$$\xi_{\mathrm{synchr}} = -\frac{2 + \langle(1-2n)\psi\rangle}{2}\,\Gamma \simeq -2\Gamma \tag{3}$$

where $\langle\ \rangle$ denotes an average over one magnetic period and ψ is the periodic solution of the equation

$$\frac{d^2\psi}{d\vartheta^2} + (1-n)\psi = 1 \tag{4}$$

For weak focusing, we note that

$$\psi \equiv 1/(1-n) \tag{4a}$$

For a strong focusing accelerator, the expression for ξ_z remains unchanged and

$$\xi_{\rho\,\mathrm{bet}} \simeq \Gamma/2 \tag{5}$$

In this case, build-up of the radial betatron oscillations will occur and this can be serious when the electrons have energies of the order of several BeV.

We have already proposed some methods[2,3] which should prevent the build-up of these radial betatron oscillations ρ_{bet} by providing additional damping. Three methods, in particular will be discussed here:

(a) arranging that the amplitude of the accelerating voltage has a radial dependence $V_0 = V_0(\rho)$;
(b) the provision of artificial coupling between the radial and vertical oscillations;
(c) the use of special 'damping' magnets.

This paper presents results which make it possible to estimate how effective these methods will be.

2. SUPPRESSION BY MODIFICATION OF ACCELERATING VOLTAGE

Method (a), which makes use of a radial voltage dependence, is not connected directly with radiation but it can be applied effectively to large electron synchrotrons where the accelerating voltage required to compensate for the radiation loss becomes very high.

By analogy with the magnetic field exponent, n, it is convenient to introduce the concept of an accelerating voltage exponent n_v.

$$n_v = -\frac{1}{V_0 k}\frac{\partial V_0}{\partial \rho}, \quad V_0 = V_0|_{\rho=0}(1 - n_v k\rho + \ldots) \tag{6}$$

where k is the equilibrium orbit curvature at a given point. Calculations show that the exponent n_v gives rise to an additional change in the amplitudes of the betatron and synchrotron oscillations associated with the decrements

$$\xi_{\rho\,\mathrm{bet}} = -\tfrac{1}{2}\langle \Gamma n_v \psi k\rangle, \qquad \xi_{\mathrm{synchr}} = \tfrac{1}{2}\langle \Gamma n_v \psi k\rangle \tag{7}$$

* Translated from *Atomnaya Energiya* **5**, 554 (1958).

It follows therefore that the damping of the betatron oscillations associated with $V_0(\rho)$ is accompanied by an equivalent build-up of the synchrotron oscillations and vice versa. One should bear in mind that for the normal strong focusing accelerator $|\xi_{\text{synchr}}| \simeq 4\xi_{\text{bet}}$ i.e. the damping of the synchrotron oscillations is considerably stronger than the build-up of the radial betatron oscillations. To eliminate radiation build-up of the radial betatron oscillations, it is necessary that ξ_{bet} in equation (7) should reach a value of about unity. The exponent n_v should therefore satisfy the condition.

$$n_v \approx n_{\text{eff}}, \qquad n_{\text{eff}} = 1 - 1/\langle\psi\rangle \tag{8}$$

For the centre of the stability region $n_{\text{eff}} \simeq |n|/5$ and in an accelerator with a magnetic field having axial symmetry $n_{\text{eff}} \equiv n$.

The above condition for n_v implies that the variation of V_0 with radius (ΔV_0) over the operating region of the ring must be sufficiently large: $\Delta V_0 \simeq V_0$. The advantage of this method is that it requires modification only to the high frequency accelerator system and leaves the bulk of the accelerator, i.e. the magnet, unaffected.

3. SUPPRESSION BY COUPLING OF RADIAL AND VERTICAL OSCILLATIONS

Method (b) relies upon the use of a sufficiently strong degree of coupling between the radial and vertical betatron oscillations. In any periodic accelerator system (without coupling) these oscillations die out with exactly the same law. For a given set of conditions, such coupling weakens the damping but there is a simultaneous reduction in the build-up of the radial oscillations (ρ_{bet}). In order to use this method it is necessary to choose an operating point which is close to a difference resonance line.

$$\mu_\rho - \mu_z = 2q\pi/N \tag{9}$$

where $N\mu_\rho$ and $N\mu_z$ are the phase changes of the oscillation during the time a particle takes to complete a circuit, N is the number of magnet periods in the accelerator, and q is an integer. If co-ordinate and velocity coupling is present, the equations for betatron oscillations can be expressed with sufficient accuracy in the form (for simplicity we consider $k = \text{const}$);

$$\ddot{\rho} + (\Gamma/\omega)(1 - \psi)\dot{\rho} + (1 - n)\rho - \Delta n\rho + \gamma z + \tau\dot{z} = 0 \tag{10}$$

$$\ddot{z} + (\Gamma/\omega)\dot{z} + nz + \Delta nz + \gamma\rho - \tau\dot{\rho} = 0 \tag{11}$$

where Δn is the perturbation which causes the departure from the exact resonance condition (9); γ and τ are the coupling parameters relating to co-ordinates and velocity, respectively.

The coupling of the ρ- and z-oscillations, determined by the parameter γ, can be effected by using quadrupole lenses at some azimuthal position. Alternatively, at the expense of a certain amount of distortion of the curvature of the guiding magnets, they can be placed in the plane perpendicular to the orbit of the particles. Coupling corresponding to the parameter τ can be achieved at the expense of the additional azimuthal field in the straight sections of the accelerator but this type of coupling turns out to be far less effective than co-ordinate coupling.

Using perturbation theory to solve equations (10) and (11), we obtain the following expressions for the damping decrements

$$\xi_{\rho\,\text{bet}} = -\tfrac{1}{4}\Gamma\{\psi + (1/g)(\psi - 2)\} \tag{12}$$

$$\xi_z = -\tfrac{1}{4}\Gamma\{\psi - (1/g)(\psi - 2)\} \tag{13}$$

where g is a quantity which characterizes the degree of coupling between the oscillations

$$g = \sqrt{\frac{(\Delta\mu)^2 + \nu^2}{\Delta\mu}}\,.$$

Here, $\Delta\mu$ is the deviation from the resonance condition (9), and ν is determined by the coupling parameter γ. If F_r and F_z are the normal solutions for the zero order approximation normalized so that Wronskian is equal to $-2i$, then

$$\Delta\mu = \frac{1}{2N}\int_0^{2\pi} [\Delta n(|F_r|^2 + |F_z|^2)]d\vartheta \tag{15}$$

$$\nu \equiv \left|\frac{1}{2N}\int_0^{2\pi} \gamma F_r F_z d\vartheta\right| \tag{16}$$

In the absence of coupling, or for large deviations from resonance ($g \to 1$), we have the equations

$$\xi_{\rho\,\text{bet}} = \tfrac{1}{2}\Gamma(1 - \psi), \qquad \xi_z = -\tfrac{1}{2}\Gamma \tag{17}$$

from which, in particular, equations (1) and (5) are derived. In the presence of coupling ($g \to \infty$), and under conditions of exact resonance (9), we have

$$\xi_{\rho\,\text{bet}} = -\tfrac{1}{4}\Gamma\,\psi, \qquad \xi_z = -\tfrac{1}{4}\Gamma\,\Psi \tag{18}$$

Since in a strong focusing accelerator ψ is small $\left(\text{for instance, } \Psi \simeq \frac{5}{|n|}\right)$ equation (18) implies that in this case the amplitudes of the radial and vertical oscillations on the average remain practically constant—they neither die away nor grow. The values of the decrements under actual working conditions can be calculated using formulae (12)–(16). For this purpose,

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it is necessary to fix the tolerance Δn and to choose values for the coupling parameters γ and ν. ν is determined not only by γ but also by the azimuthal extent of the quadrupole lenses which provide the coupling.

4. SUPPRESSION BY DAMPING MAGNETS

We turn now to the question of the damping magnets. Let us consider the general case of an arbitrary magnet system in a synchrotron type of accelerator for which the equilibrium orbit has curvature k and field exponent n, both being periodic functions of the generalized azimuth angle θ.

$$k(\theta + \theta_{0k}) = k(\theta), \qquad n(\theta + \theta_{0n}) = n(\theta) \tag{19}$$

In such a system, the equations of motion of the particles in the $z = 0$ plane will have the form

$$\rho'' + (\Gamma/c)\rho' + k^2(1 - n)\rho = k\,\Delta E/E_s \tag{20}$$

$$(1 - k\rho)\frac{\Delta E'}{E^s} = \frac{eV_0 \cos\phi}{E_s L} - \\ - \frac{\Gamma}{c}\left(1 + 2\frac{\Delta E}{E_s} - 2n\rho k\right), \quad \Delta E = E - E_s \tag{21}$$

where the differentiation is carried out along the arc σ of the equilibrium orbit. The quantity Γ (see (2)) is a rapidly varying function of the argument and L_s and E_s are the equilibrium values of the orbital path L and of the energy E. We will suppose that for an equilibrium particle in the median plane the accelerating system compensates for the radiation losses. Introducing the function ψ, the periodic solution of the equation

$$\psi'' + k^2(1 - n)\psi = kk_0, \qquad k_0 = \frac{2\pi}{L_s} \tag{22}$$

we obtain the damping decrements of the synchrotron and betatron oscillations

$$\xi_{\text{synchr}} = -\frac{1}{2}\left\langle 2\Gamma\left(1 - n\frac{k}{k_0}\psi\right) + \frac{k}{k_0}\psi\Gamma\right\rangle \tag{23}$$

$$\xi_{\rho\,\text{bet}} = -\tfrac{3}{2}\langle\Gamma\rangle - \xi_{\text{synchr}} \tag{24}$$

It is easy to see that for an accelerator in which the equilibrium orbit has a constant radius of curvature and for which, consequently, Γ is a constant, these equations assume the forms (3) and (5). The case $k = k(\theta)$ for strong focusing was first considered by ROBINSON[4] and later, in a generalized form, by ORLOV and TARASOV.[6]

It follows from equations (23) and (24) that a magnet system which damps the radial betatron oscillations and which does not at the same time cause a build-up of the synchrotron oscillations must satisfy the conditions

$$-\frac{1}{2} < \frac{\langle nk^3\psi\rangle}{k_0\langle k^2\rangle} < 1 \tag{25}$$

Here the quantity $\langle\psi\rangle \ll 1$ has been neglected. The left hand inequality is the condition for stability of the betatron oscillations; the right hand inequality is the stability condition for synchrotron oscillations.

Thus, the damping of the betatron oscillations causes a further reduction of the curvature of the equilibrium orbit in the focusing magnets. If, for constructural reasons, it is desirable to have identical fields in all the guiding magnets, then it is more convenient to place special damping magnets in the straight sections of the accelerator. These can, for example, consist of two sectors with opposed magnetic fields which fall off rapidly with the distance from the centre of the accelerator.[4]

In view of the criterion (25), we must admit that our earlier suggestion[2] that damping magnets with $n \leqslant 1$ would be suitable in a strong focusing magnet is erroneous. In fact, damping magnets with such a value of n would be effective only for the case where $\Psi \simeq 1$, i.e. for weak focusing*

5. CONCLUSIONS

We note that the formulae we have derived have the peculiar feature that the total decrement of the synchrotron and betatron motion is equal to $-\frac{2}{3}\langle\Gamma\rangle$ for all cases. If we now consider vertical oscillations which have been coupled with the radial motion then this total becomes $-2\langle\Gamma\rangle$. This situation is not accidental but follows from the general theorem of Liouville-Ostrogradskii[5] according to which the Wronskian of the solution of an n-th order differential equation transforms according to

$$W = \text{const.}\,\exp\left(-\int\frac{P_{n-1}}{P_n}\,dx\right) \tag{26}$$

where P_{n-1} and P_n are the coefficients of the $(k-1)$th and kth derivatives. The coefficient $\dfrac{P_{n-1}}{P_n}$, which determines the damping Wronskian, is associated with the energy dependence of the radiation intensity which in any magnet system has the form $W \sim E^2$ and with the damping which is strictly due to radiation and which has a decrement equal to $-\Gamma/2$ for the radial

* In the reference cited,[2] the expression $\left\langle\dfrac{n}{1-n}\right\rangle$ should be replaced by $\dfrac{n_{\text{eff}}}{1 - n_{\text{eff}}}$ (see equation (8)).

and vertical oscillations. Since neither of these factors depends upon the nature of the magnet system, the accelerating apparatus, oscillation coupling etc., it must clearly follow that the damping Wronskian is also independent of these factors. In fact, calculations confirm that the coefficient $\frac{P_{n-1}}{P_n}$ always equals 3Γ for a system with two degrees of freedom (radial betatron and synchrotron oscillations) or equals 4Γ if the system has in addition a vertical degree of freedom (coupled radial and vertical oscillations).

It follows immediately therefore that any arrangement which increases the damping of one type of oscillation must certainly decrease the damping of the other type. Since the Wronskian of the solution necessarily dies away (implying a decrease of the total six dimensional phase space occupied by the oscillations) a well chosen damping system will ensure damping for all three degrees of freedom.

Apart from this general result, application of the Liouville-Ostrogradskii theorem provides a simple means of finding the damping decrement of the betatron oscillations for various versions of the damping system. To do this, it is sufficient to determine the decrement of the synchrotron oscillations, a procedure which is very much simpler.

A similar approach can also be applied to other effects where the rate of energy loss is associated with the energy of the particle: for example, the case of bremsstrahlung radiation due to the nuclei present in the residual gas.

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LETTERS TO THE EDITOR

A resonance method for localizing and heating a plasma by means of a varying electromagnetic pressure*

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HEREIN we discuss briefly the resonance excitation of volume oscillations of a plasma bunch by an amplitude-modulated electromagnetic field which produces a uniformly distributed pulsating pressure on the surface of the plasma. This problem has relevance to attempts to achieve pulsed high temperatures and pressures in a plasma such as are necessary, for instance, for the intensification or initiation of thermonuclear processes when the amplitude of the attainable electromagnetic field is limited in magnitude. A modulated isotropic pressure on the surface of a quasi-spherical bunch can be realized by subjecting the plasma to an intense modulated short wave radiation field, the modulation being produced by a magnetic field which varies rapidly and randomly in the neighbourhood of the bunch, produced either by a modulated superposition of three rapidly varying magnetic fields or (for a rapidly moving bunch) by a spatially modulated magnetic field which is randomly distributed along the path of the bunch, etc.

For simplicity we assume that the alternating electromagnetic field is screened by the plasma electrons and does not penetrate into the interior of the plasma. On the surface of the plasma there will then be a pressure $P_{av}(t) \simeq H^2_{av}(t)/8\pi$, where H_{av} is the time average of the magnetic field intensity on the plasma surface.

The excitation produced by the alternating pressure depends essentially on the ratio between the effective modulation frequency and the characteristic pulsation frequency of the bunch. Volume oscillations will be excited strongly and rapidly if the effective modulation frequency is close to the quasi-acoustical resonance frequency ν_{res} of radial pulsations of the bunch. For a dense plasma the order of magnitude of ν_{res} is given by the ratio of the volume wave velocity in the plasma to the dimensions of the bunch: $\nu_{res} \approx u/2a_0 \sim (\varepsilon_T/A)^{\frac{1}{2}}/a_0$ Mc/s, where ε_T is the thermal kinetic energy of the plasma ions, A their atomic weight, a_0 the radius of the bunch; for instance, for $\varepsilon_T \approx 30$ eV and $a_0 \approx 10$ cm, $\nu_{res} \lesssim 1$ Mc/s, which demonstrates the practical possibility of exciting resonance pulsations.

We are assuming that the main mass of plasma in the bunch takes part in the pulsation simultaneously (as for instance, in a uniform deformation, provided the rate of variation of external pressure is not too great). Then the equation for the excitation of forced pulsations has the form (in dimensionless units) $(1/z^2) \cdot d^2z/d\tau^2 = 1/(z^{3\gamma}) - f_{ext}(\tau)$, where $z = a(t)/a_0$, $f = P_{ext}/P_0$ and $\tau \sim \nu_{res} \cdot t$.

This equation can be solved numerically for various forms of the modulation function $f(\tau)$ (for comparison one can point to a similar problem concerning the excitation of pulsations of a gas-bubble in a liquid by ultrasonic or shock waves). If $f(\tau)$ is a periodic function the oscillations of the bunch are in the nature of beats, since the frequency of the pulsations depends on the amplitude of the oscillations, and at regular intervals the oscillations are suppressed by the external force. Therefore, when one wishes to obtain sufficiently large oscillation amplitudes it is desirable to vary the period of the external excitation as the period of the pulsations varies. Such a synchronization can be arranged for in advance, or else it can be ensured by a device governed by the pulsations of the bunch.

Owing to the small surface area of the bunch at the moment of maximum compression, and owing to the rapidity with which oscillations are excited, the development of instability effects is negligible and the influence of the evaporation of plasma particles is diminished. Evaporation can occur because the basic force causing the bunching, which is the dynamical pressure of the radially pulsating plasma does not remove the possibility of particles leaving the surface. This effect can be diminished by the use of incomplete modulation of the amplitude of the external field. This justifies the discussion of methods of effective rapid excitation of strong pulsations by small changes of a large (maximum obtainable) pressure.

We estimate the order of magnitude of the transformation coefficient of pressure, i.e. the ratio of the maximum external pressure P_{ext} used, to the maximum obtainable internal pressure in dense plasma. We make the rough assumption that the process is an ideal one. Equating the work done by the external pressure to the change of internal energy of the gas (at the instants of maximum contraction and expansion the kinetic energy is zero) we obtain

$$P_{ext}(V_{max} - V_{min}) \simeq \frac{1}{\gamma - 1}(P_{max}V_{min} - P_1V_{max}),$$

where P_1 is the internal pressure at the moment of greatest expansion, γ the ratio of specific heats for the plasma gas and V the volume of the bunch. But we have $P_1V^{\gamma}_{max} = P_{max}V^{\gamma}_{min}$ so that

$$K_P = \frac{P_{max}}{P_{ext}} = (\gamma - 1)\frac{(K_V - 1)}{\left(1 - \frac{1}{K_V^{\gamma-1}}\right)},$$

where $K_V = V_{max}/V_{min}$. We assume that $K_V \gg 1$, e.g. $K_V \sim 30$, then $K_P \simeq (\gamma - 1)K_V \sim 20$ for $\gamma = 5/3$ (monatomic gas).

In addition to mass pulsations it is of interest to discuss a sharply pulsed or a high frequency modulation of the pressure which leads either to the production of convergent spherical waves of high amplitude, or to the excitation of resonances in the higher harmonics and to a strong heating of the plasma due to the dissipation of these high frequency acoustical oscillations.

It is evident that a similar discussion of the excitation of quasi-acoustical resonance oscillations may be performed for plasma bunches which are prevented from dispersal in certain directions by a strong magnetic field and also for elongated forms of plasma bunches for instance for a plane layer, a cylinder or a torus of plasma; on their surfaces one can utilize the action of pulsating currents produced by varying electric fields. However, elongated configurations are unstable against bending,

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therefore spherical symmetry seems to be more promising, since it retains its shape longer.

One should note that in dynamical conditions of pulsation the stability conditions for plasma configurations may differ from those of the stationary case and that in certain cases the formation of dynamically stable shapes might be possible.

We note also that in the case of a less dense or strongly heated plasma (when ion collisions can be neglected) the rapid application of an electromagnetic interaction to the electrons of the quasi-neutral plasma will cause an intense radial oscillation of the ions, due to the Coulomb interaction of electrons and ions. The ions will repeatedly pass close to the centre at high relative velocities. This method of 'super-thermal' excitation of ion oscillations has a curious analogy to the creation of 'balls of yarn' from charged particles in constant electric fields.

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On the behaviour of small plasma bunches in a waveguide and their interaction with the conducting walls*

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Methods have been suggested by Veksler[1] for the radiative acceleration of plasma bunches. It is of interest in this connexion to elucidate the conditions ensuring the detachment and repulsion of quasi-neutral plasma bunches from the walls of a waveguide in order to achieve an increase of radiative or other dissipative properties and to ensure the Coulomb trapping of ions by electrons. These conditions of detachment are essential also in the discussion of various methods for injecting and accumulating plasma and for transporting plasma bunches carrying magnetic fields through pipes. In the work by M. L. Levin and Askaryan,[2] the radial stability of small bunches near the axis of a waveguide has been investigated. In the same paper, the axial stability and localization of plasma bunches moving through the neighbourhood of the axis of extended magnetic fields are also examined.

In the present article we evaluate the radial force acting on a small plasma bunch in the simplest types of wave field when the bunch is displaced an arbitrary distance greater than its dimensions; an estimate is also made of the strength of the interaction of the bunch with the conducting walls and it is pointed out also that there exist several possible types of pipes and reflectors for current-carrying bunches.

In discussing the behaviour of a bunch in a waveguide, one should bear in mind that for sufficiently small induced dipole moments of the bunch (for instance for very small dimensions of the bunch or small dynamical polarizability) the interaction of the bunch with the walls is small compared with the effect of the wave field, even in the vicinity of the walls. This follows from the fact that the interaction with the wave is proportional to the first power of the small dipole moment, while the interaction with the wall is proportional to its square. For such small bunches one can evaluate the radial forces at any displacement from the axis.

In a unified notation the expressions for axially symmetric H_0 and E_0 waves have the form (the left hand terms in the curly brackets refer to the H_0 wave):

$$\{H_z;\ E_z\} = \{H_a;\ E_a\}J_0(\kappa r)\sin(\omega t - hz),$$

$$\{H_r;\ E_r\} = \{H_a;\ E_a\}\frac{h}{\kappa}J_1(\kappa r)\cos(\omega t - hz),$$

$$\{E_\varphi;\ H_\varphi\} = \{-H_a;E_a\}\ \frac{k}{\kappa}J_1(\kappa r)\cos(\omega t - hz).$$

Here H_a and E_a are the amplitudes of the waves on the axis of the waveguide, r is the distance from the axis, κ and h are the transverse and longitudinal wave numbers ($h^2 + \kappa^2 = k^2$). These wave numbers are related to the radius of the waveguide R by the boundary condition $J_1(\kappa R) = 0$, ($\kappa \simeq 3{\cdot}8/R$) for the H_0 wave and $J_0(\kappa R) = 0$ ($\kappa \simeq 2{\cdot}4/R$) for the E_0 wave.

In the H_0 wave the time averages of radial forces acting on the electric (P) and magnetic (M) moments of the bunch are given by the expressions:

$$(F_P)_{\text{av}} = \left(P_\varphi \frac{\partial E_\varphi}{\partial r}\right)_{\text{av}} = \frac{\alpha}{2}H_a^2\frac{k^2}{\kappa^2}J_1J_1'\cos\varphi_P,$$

$$(F_M)_{\text{av}} = \left(M_r\frac{\partial H_r}{\partial r} + M_z\frac{\partial H_z}{\partial r}\right)_{\text{av}}$$

$$= \frac{\beta}{2}H_a^2\left\{\frac{h^2}{\kappa^2}J_1J_1' + J_0J_0'\right\}\cos\varphi_M.$$

In these expressions α and β are the coefficients of dynamic, electric and magnetic polarizability for an isotropic bunch and φ_M and φ_P are the phase shifts between the electric and magnetic moments of the bunch and the field (in particular for a screening quasi-spherical bunch of radius a we have $\alpha = -2\beta = a^3$ and $\varphi_P = \varphi_M = 0$) J and J' are a Bessel function of the argument κ and its derivative with respect to r. (In the following we shall use the well known relations $J_0' = -\kappa J_1$ and $J_1' = -J_1/r + \kappa J_0$.)

The total radial force acting on the bunch is

$$(F_r)_{\text{av}} = -\frac{1}{2}H_a^2\left\{\alpha\frac{k^2}{\kappa^2}J_1J_1'\cos\varphi_P + \beta\left(\frac{h^2}{\kappa^2}J_1J_1' + J_0J_0'\right)\cos\varphi_M\right\}.$$

Near the axis we have $J_0J_0' = -\frac{1}{2}\kappa^2 r$, $J_1J_1' = \frac{1}{4}\kappa^2 r$, therefore the condition for axial stability has the form

$$\alpha k^2\cos\varphi_P + \beta(k^2 - 3\kappa^2)\cos\varphi_M < 0.$$

Near the walls $J_0J_0' = \kappa^2J_0^2\rho$ and $J_1J_1' = -\kappa^2J_0^2\rho$ ($\rho = R - r$), therefore the condition for detachment of the bunch from the wall by the wave field is

$$\alpha k^2\cos\varphi_P + \beta(k^2 - 2\kappa^2)\cos\varphi_M > 0.$$

For instance, while a bunch in dense plasma is unstable relative to the axis of an H_0 wave the field of this wave in the neighbourhood of the wall of the waveguide does effect the detachment of the cluster from the wall by acting on it with the force $(F_r)_{\text{av}} = -\frac{1}{2}a^3H_a^2J_0^2(\frac{1}{3}k^2 + \kappa^2)\rho$. The analysis of the character of the force distribution shows that within the waveguide there is an equilibrium radius about which the position of the bunch is stable. The bunch is compressed by the magnetic wave field, the vector of which alternates or rotates as a result of the phase shift between the components H_r and H_z.

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Various combinations of wave fields and constant external magnetic fields also are possible which have such values along the flight path of the bunch that in the co-ordinate frame of the bunch the magnetic field wanders, rotates or changes its direction and by this means makes possible the localization and stability of the moving plasma bunch.

We estimate the strength of the interaction of a bunch with the conducting wall, assuming that the distance between the wall and the bunch is greater than the dimensions of the latter, but is considerably less than the dimensions of the waveguide. If the conductivity of the wall is sufficiently high the field of the surface charges and currents that are produced when the polarized bunch approaches the wall may be replaced by an image field. The longitudinal and transverse components of the magnetic image of a varying or moving magnetic dipole are related to the components of the magnetic moment of the dipole itself by the relation $M_{\parallel}' = M_{\parallel}$ and $M_{\perp}' = -M_{\perp}$ while for an electric dipole the image vector satisfies $P'_{\parallel} = -P_{\parallel}$ and $P_{\perp}' = P_{\perp}$. The component normal to the surface of the force of interaction between the bunch and the walls is

$$F_n = \frac{3}{16\rho^4}\{2(M_{\perp}^2 - P_{\perp}^2) + (M_{\parallel}^2 - P_{\parallel}^2)\}_{\text{av}}$$

In the H_0 wave only the longitudinal component H_z and the longitudinal component of magnetic moment are non-zero near the walls, so that $F_n = \frac{3}{32\rho^4}\beta^2 H_a^2 J_0^2$. This force of repulsion increases rapidly as the bunch approaches the wall. Comparing the interaction forces of the wave and the interaction with the wall for a dense bunch one sees that if $\rho > a[a^2(\frac{1}{2}k^2 + \kappa^2)]^{-\frac{1}{3}}$ the displacing force due to the field exceeds the force of repulsion from the wall.

We now discuss the behaviour of a small bunch in the field of an E_0 wave. The average radial forces acting on the bunch are

$$(F_P)_{\text{av}} = \left(P_z\frac{\partial E_z}{\partial r} + P_r\frac{\partial E_r}{\partial r}\right)_{\text{av}} = \frac{\alpha}{2}E_a^2\left(J_0J_0' + \frac{h^2}{\kappa^2}J_1J_1'\right)\cos\varphi_P,$$

$$(F_M)_{\text{av}} = \left(M_\varphi\frac{\partial H_\varphi}{\partial r}\right)_{\text{av}} = \frac{\beta}{2}E_0^2\frac{k^2}{\kappa^2}J_1J_1'\cos\varphi_M$$

The total radial force is

$$(F_r)_{\text{av}} = \frac{1}{2}E_a^2\left\{\alpha\left(J_0J_0' + \frac{h^2}{\kappa^2}J_1J_1'\right)\cos\varphi_P + \beta\frac{k^2}{\kappa^2}J_1J_1'\cos\varphi_M\right\},$$

and thus the condition for stability near the axis has the form

$$\alpha(k^2 - 3\kappa^2)\cos\varphi_P + \beta k^2\cos\varphi_M < 0.$$

Near the walls of the waveguide $J_1' = -J_1^2/R \gg J_0'$ and therefore the wave field will keep the bunch off the walls under the condition that

$$\alpha h^2\cos\varphi_P + \beta k^2\cos\varphi_M > 0.$$

In particular for a dense plasma bunch the force of interaction with the field is

$$(F_r)_{\text{av}} = -\frac{a^3}{4}J_1^2\frac{1}{\kappa^2 R}E_a^2(k^2 - 2\kappa^2)$$

and the detachment of the bunch from the wall will occur if $k^2 > 2\kappa^2$. The magnitude of the force effecting the detachment from the wall remains finite even for small distances from the wall.

The force of close interaction of bunch and wall is determined by the presence of the electric radial and magnetic azimuthal moments whose amplitudes are $P_r = \alpha E_a hJ_1/\kappa$ and $M_\varphi = \beta E_a kJ_1/\kappa$, therefore the normal component of the force of interaction with the wall is

$$F_n = \frac{3}{32\rho^4}E_a^2J_1^2\frac{1}{\kappa^2}(\beta^2k^2 - 2\alpha^2h^2),$$

i.e. repulsion from the wall occurs if $\beta^2k^2 > 2\alpha^2h^2$. Neglecting the differences of the squares of the wave numbers for a dense bunch we obtain $F_r/F_n \sim (\rho/R)(\rho/a)^3$ thus if $\rho \gg a$ the interaction with the wall can be neglected.

It is easy to predict the effect on the processes discussed of applying a strong constant magnetic field which effectively 'freezes in' all the polarizabilities of the bunch other than the longitudinal electric polarizability. However, for brevity we shall not dwell on these and similar effects connected with the non-isotropy of the medium or the shape of the bunch.

In addition to the high-frequency magnetic moment induced by the wave field the bunch may have its own slowly varying magnetic moment produced, for instance, by residual or artificially maintained circulating currents (the attraction of such currents weakens the longitudinal dispersal of the bunch) or by the plasma diamagnetism in an external constant field. This intrinsic moment does not influence the time average of the interaction of the wave field with the polarization of the bunch and therefore the possibilities of independently realizing localization and transverse stability of a bunch are improved. However, the field of such a moment may essentially increase the force of repulsion between the bunch and the wall as it approaches the wall. Therefore a channel with conducting walls may serve as a 'plasma conductor' for a flux of plasma bunches carrying magnetic fields. We mention tentatively yet another example of the interaction of current-carrying plasma bunches with conducting walls: the localization of an approaching current-carrying bunch by concave metallic surfaces. The induction currents arising on the surface of such mirrors will not only act as brakes on the approaching bunch and inductively reinforce its current, but they will contribute also to its radial compression (or weaken its dispersal). Owing to the high concentration of magnetic lines of force surrounding the ideally conducting surface of a mirror, the reflection of a current-carrying bunch by a concave mirror will be more effective than by a plane mirror. By choosing the shape of the concave surface and the conditions for the path of the bunch one can vary the dynamical reflection within wide limits and in particular make the instants of reflection and maximum contraction coincide. Such concentrating reflectors may be used for pulsed localization and heating of moving plasma.

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A possible non-stationary thermonuclear reactor*

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THE energy generated in a stationary gas discharge can be removed only in the form of heat. Therefore it is of interest to investigate non-stationary models that allow a direct transformation of a part of the nuclear energy of a plasma into electrical energy.

The model proposed consists of a cylindrical or spheroidal plasma column performing forced radial oscillations under the influence of an external alternating axial magnetic field. In this arrangement the temperature of the plasma will oscillate around T_e, the temperature at which the energy generated by nuclear reactions compensates for the radiation losses. In the following we derive a condition for the periodicity of the oscillations, and show that for this condition to be satisfied the nuclear energy generated in one cycle must exceed considerably the radiative energy. The difference of energies must be passed to the external winding producing the magnetic field.

It has been shown[1] that a plasma cylinder compressed by a magnetic field with straight lines of force is dynamically stable.

Thus it would be more sensible to study it during times which are greater than those of its own oscillations. The temperature T_e necessary for operating a thermonuclear reactor using deuterium plasma is of order of magnitude 10^{8}°K so that the speed of sound in the plasma $\sim 10^{8}$ cm. sec^{-1}. Hence the proper frequency of a column of about 10 cm diameter is about 10^{7} c/s. We shall assume that the frequency with which the external magnetic field varies has the order of ordinary a.c. frequencies, $\sim 10^{3}$ c/s. We can also replace the equation of motion by a condition of equilibrium because the plasma oscillations under the action of a magnetic field are very slow compared to the proper oscillations. In the following we shall confine ourselves to the case when the magnetic field H outside the plasma differs little from the external field H_e:

$$H = H_e(1 - h),$$

where $h \ll 1$.

In addition we shall assume that the temperature at each instant is the same in the whole column. Under these assumptions the hydromagnetic equations have the following form

$$\frac{\partial}{\partial r}\left[(1 + Z)n_i kT + \frac{H_e^2(1 - 2h)}{8\pi}\right] = 0, \tag{1}$$

$$\frac{\partial H_e}{\partial t} + vH_e\frac{\partial h}{\partial t} + H_e(1 - h)\frac{\partial (r,v)}{r\,\partial r} = -\frac{c^2}{4\pi\sigma}H_e\frac{\partial}{r\,\partial r}\left(r\frac{\partial h}{\partial r}\right), \tag{2}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{r\partial r}(rvn_i) = 0. \tag{3}$$

The energy equation should be given for the column as a whole

$$\int_0^{r^*}\left[-\frac{3}{2}n_i kT\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\ln Tn_i^{-2/3} - n_i^2(\varepsilon_r - \varepsilon_b) + \frac{c^2}{16\pi^2\sigma}H_e^2\left(\frac{\partial h}{\partial r}\right)^2\right]2\pi r\,dr = 0. \tag{4}$$

Here n_i is the ion density, Z their charge, σ the electric conductivity of the plasma, $r^*(t)$ the radius of the column, and ε_b and ε_r are quantities depending on temperature in those terms which take into account the losses due to bremsstrahlung and the nuclear energy generated (omitting that part of the energy which is carried away by neutrons).

This system of equations must satisfy the following boundary conditions

$$n_i = 0 \quad \text{and} \quad h = 0 \quad \text{for} \quad r = r^* \tag{5}$$

Equations (1), (2) and (3) have simple solutions to the accuracy required; we shall use these in the following:

$$n_i = \frac{2N_i}{\pi r^{*2}}(1 - r^2/r^{*2}), \quad v = \frac{r}{r^*}\frac{dr^*}{dt}; \qquad h = \frac{(1 + Z)N_i kT}{H_e^2 r^{*2}}(1 - r^2/r^{*2}); \tag{6}$$

$$\frac{\partial}{\partial t}(H_e r^{*2})^2 = -\frac{2c^2}{\pi\sigma}\cdot(1 + Z)N_i kT. \tag{6a}$$

Here N_i is the number of ions in 1 cm of plasma column length and T is an as yet arbitrary function of time. The temperature will be determined later from the energy balance equation so that $r^*(t)$ is determined by equation (6a).

It follows from this equation that the time constant for the dissipation of magnetic energy is about 1 sec. As we are considering plasma oscillations of a considerably smaller period (about 10^{-2} sec) we can in the first approximation neglect energy dissipation. Then

$$(H_e r^*)^2 = \Phi = \text{const.} \tag{7}$$

Inserting expressions (6) and (7) into equation (4) we obtain

$$\frac{d}{dt}\frac{T}{H^{2/3}} - \frac{8N_i}{9\pi(1 + Z)\Phi kt}H\frac{T}{H^{2/3}}(\varepsilon_r - \varepsilon_b) = 0. \tag{8}$$

For the simplification of further calculations we expand the difference $\varepsilon_r - \varepsilon_b$ into a series around the temperature T_e and leave only the first term

$$\varepsilon_r - \varepsilon_b \simeq m(T - T_e). \tag{9}$$

Let t^* be the oscillation period of the magnetic field and H_m be its mean value. We introduce the notation

$$\tau = t/t^*; \quad H_e = H_m(1 + \alpha); \qquad \frac{T}{T_e}\left(\frac{H_m}{H_e}\right)^{2/3} = u; \quad \frac{8mN_i t^* H_m}{9\pi(1 + Z)k\Phi} = \kappa, \tag{10}$$

and rewrite equation (8) in the form

$$\frac{du}{d\tau} = \kappa(1 + \alpha)^{1/3}[u(1 + \alpha)^{2/3} - 1]. \tag{11}$$

One should integrate this equation setting $d = \text{const.} \cos 2\pi$, but this leads to too lengthy calculations. For simplification we shall assume that the field varies according to the law

$$1 + \alpha = 1 + \alpha_0 \quad \text{for} \quad 0 < \tau < \tfrac{1}{2}$$
$$1 + \alpha = 1 - \alpha_0 \quad \text{for} \quad \tfrac{1}{2} < \tau < 1 \tag{12}$$

where $\alpha_0 = \text{const.}$ This simplification affects the final results only slightly although it contradicts the initial assumption that the field varies slowly.

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A solution of equation (11) must be found in the form of a continuous function of period 1. It can be written in the following form

$$u = (1+\alpha_0)^{-2/3} + [(1-\alpha_0)^{-2/3} - (1+\alpha_0)^{-2/3}]\frac{e^{\kappa(1-\alpha_0)}-1}{e^{\kappa}-1}\,e^{\kappa(1-\alpha_0)\tau} \quad \text{for } 0<\tau<\tfrac{1}{2}$$

$$u = (1-\alpha_0)^{-2/3} - [(1-\alpha_0)^{-2/3} - (1+\alpha_0)^{-2/3}]\frac{e^{\frac{\kappa(1+\alpha_0)}{2}}-1}{e^{\frac{\kappa(1-\alpha_0)}{2}}(e^{\kappa}-1)}\,e^{\kappa(1-\alpha_0)\tau} \quad \text{for } \tfrac{1}{2}<\tau<1. \tag{13}$$

Evidently, if the condition of periodicity is to be satisfied the plasma parameters at this point in the cycle cannot be given arbitrarily. The relation between them is determined by solution (13).

We can now evaluate the excess energy Δ produced in a cycle. In 1 cm of column length it is

$$\Delta = \int_0^{t^*} dt \int_0^{r^*} n_i^2 m(T-T_e)2\pi r\,dr. \tag{14}$$

Inserting here solution (13) we get

$$\Delta = \left[\left(\frac{1+\alpha_0}{1-\alpha_0}\right)^{2/3} + \left(\frac{1-\alpha_0}{1+\alpha_0}\right)^{2/3} - 2\right] \times \frac{\left(e^{\frac{\kappa}{2}(1+\alpha_0)}-1\right)\left(e^{\frac{\kappa}{2}(1-\alpha_0)}-1\right)}{e^{\kappa}-1}\,\frac{3}{2}(1+Z)N_i kT_e. \tag{15}$$

One sees that Δ is always positive, i.e. the work done by the plasma against the forces of the magnetic field is on the average positive, and energy passes from the plasma to the electric network. If one can achieve a situation such that $\frac{1}{2}\kappa(1-\alpha_0)$ is less than 2 or 3 (which is possible if one uses very strong magnetic fields) the excess energy over a cycle will depend only slightly on the other parameters and will be equal in order of magnitude to the mean energy of the plasma. The electric energy obtained in unit time will be proportional to the frequency. For $\kappa \ll 1$ the power generated will not depend on frequency.

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The equilibrium distribution of current density in straight high-current discharges†

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IN this paper we derive the equilibrium distribution of electronic and ionic densities in a straight discharge in the presence of an additional current flowing along the axis of the discharge chamber.

We assume that in the chamber, which is formed by two coaxial cylinders of radii r_1 and r_2 ($r_1 < r_2$), there is a discharge current J and that along the axis of the chamber there is an additional conductor carrying a current I. We denote the electronic and ionic densities by $\rho_e(r)$ and $\rho_i(r)$ respectively and the corresponding temperatures by T_e and T_i. We also denote by β_e and β_i* the respective ratios of the longitudinal drift velocities of electrons and ions to the velocity of light. If one then assumes that the pressure tensor is isotropic and that the ions are singly charged the equations determining the stationary distribution take on the form†

$$\left.\begin{aligned} \rho_e[E_r - \beta_e H_\phi] + \frac{k}{e}\frac{\partial}{\partial r}(\rho_e T_e) = 0; \\ \rho_i[E_r - \beta_i H_\phi] - \frac{k}{e}\frac{\partial}{\partial r}(\rho_i T_i) = 0; \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r} rE_r = 4\pi e(\rho_i - \rho_e); \\ \frac{1}{r}\frac{\partial}{\partial r} rH_\phi = 4\pi e(\rho_i\beta_i - \rho_e\beta_e) \end{aligned}\right\} \tag{2}$$

Here k is Boltzmann's constant and e the absolute value of the electron charge.

We assume that the drift velocities β_e and β_i and the temperatures T_e and T_i are independent of the radius; this appears to be completely correct for not very high densities. One can then show that if one assumes that it is possible for a charge q to form on the axis, given by

$$q = \frac{I}{c}\frac{T_i\beta_e + T_e\beta_i}{T_e + T_i} \tag{3}$$

the non-zero solution of equations (1) and (2) is

$$\left.\begin{aligned} \rho_e(r) = \tilde{\rho}_e \frac{r^p}{(1+\xi^{-1}r^{p+2})^2}; \\ \rho_i(r) = \tilde{\rho}_i \frac{r^p}{(1+\xi^{-1}r^{p+2})^2}; \end{aligned}\right\} \tag{4}$$

where

$$\left.\begin{aligned} \tilde{\rho}_e = \frac{k}{2\pi e^2}\frac{(p+2)^2}{\xi}\frac{T_e(1-\beta_i^2) + T_i(1-\beta_e\beta_i)}{(\beta_e-\beta_i)^2}; \\ \tilde{\rho}_i = \frac{k}{2\pi e^2}\frac{(p+2)^2}{\xi}\frac{T_e(1-\beta_e\beta_i) + T_i(1-\beta_e^2)}{(\beta_e-\beta_i)^2}; \end{aligned}\right\} \tag{5}$$

and

$$p = \frac{4}{x-1}\left[1 - \frac{I}{J_0}(1+x)\right]; \tag{6}$$

$$x = \tfrac{1}{2}[(z-1)(\gamma-1) + \sqrt{[(z-1)^2(\gamma-1)^2 + 4z]}] \tag{7}$$

$$\xi = xr_1^{p+2}; \quad z = \left(\frac{r_1}{r_2}\right)^{p+2}; \quad \gamma = \frac{J_0 - 2I}{J};$$

$$J_0 = \frac{2kc}{e}\frac{T_e + T_i}{\beta_i - \beta_e}.$$

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† Translated from *Atomnaya Energiya* **5**, 648 (1958).

* We note that in all the formulae β_e and β_i denote the algebraic values of the velocities, i.e. if we assume that $J > 0$ then $\beta_i > 0$ and $\beta_e < 0$.

† We disregard processes occurring in the neighbourhood of the electrode.

Thus the introduction of the additional axial current I has the result that a stationary state becomes possible only in the presence of a charge q on the axis; this charge is zero only when $I = 0$ or $\beta_i/\beta_e = -T_i/T_e$. It follows from the expressions (4) and (5) that the gas column is charged as a whole and that the charge per unit length is

$$Q = \frac{J}{c}\frac{T_e\beta_i + T_i\beta_e}{T_e + T_i} = q\frac{J}{I} \tag{8}$$

Taking into account the relations (4) and (5) we obtain for the current density

$$j(r) = \frac{J_0}{\pi}\frac{(p+2)^2}{4\xi}\frac{r^p}{(1+\xi^{-1}r^{p+2})^2} \tag{9}$$

Hence it follows that with increasing discharge current its distribution across the section changes.

The current density reaches a maximum value at $r = r_m$ where

$$r_m = r_1\left(\frac{px}{p+4}\right)^{\frac{1}{p+2}}, \tag{10}$$

and

$$j(r_m) = \frac{J_0}{\pi}\frac{p(p+4)}{16r_m^2} \tag{11}$$

We discuss some interesting cases in more detail.

1. $I = 0$.

In this case it follows from (6) and (10) that the maximum of current density is always on the inner wall ($r_m = r_i$). The nature of its distribution across the section is determined by the quantity p. For weak current ($J \ll J_0$) the index p is near to zero and the current density is almost independent of radius. With increasing current p increases and a greater part of the current begins to flow in the neighbourhood of the internal chamber walls. Finally for high current ($J \gg J_0$) we obtain

$$q = 4\frac{J}{J_0}; \qquad j(r_m) = \frac{J^2}{\pi r_1^2 J_0} \tag{12}$$

i.e. practically the whole current is concentrated in a narrow region around $r = r_1$. If one assumes that $r_1 = 0$ the distribution takes on a delta function character even for $J = J_0$. In the particular case $r_1 = 0$ one gets $r_m = 0$ and $T_i = 0$; this result was obtained by THONEMANN and COWHIG.[1]

2. $I = \text{const.} < 0$ (I and J opposite in direction).

For small currents ($J \ll J_0$) the index p has its smallest value, which is $-4I/J_0$, and the current density increases monotonically with increasing radius. With increasing current p increases and the dependence of current density on radius becomes more and more pronounced. From the current value $J = -I$ onwards the maximum of density moves away from the external wall of the chamber and is displaced in the direction towards the internal wall. For $J \gg J_0 - 2I$ the nature of the current density distribution becomes similar to that in the case $I = 0$.

In conclusion we consider the case in which the central conductor is a 'return lead' and carries a certain part of the discharge current,

$$I = -kJ.$$

In this case the current maximum always lies inside the chamber and with k varying from zero to 1 it moves from the internal wall ($r_m = r_1$) to the external wall ($r_m = r_2$). For small currents ($J \ll J_0$) p is close to zero and the distribution is almost uniform across the section only having a weak maximum at

$$r = [r_1^2 + k(r_2^2 - r_1^2)]^{1/2}.$$

With increasing current, r_m decreases somewhat, the index p increases and the maximum of the distribution becomes more and more pronounced. Finally for $J \gg J_0/(2k)$

$$\gamma \approx 2k, \qquad p \approx \frac{4k(x+1)}{x-1}\frac{J}{J_0}$$

and the whole current is concentrated in a narrow region in the vicinity of

$$r_m = r_1 x^{\frac{1}{p+2}}.$$

Thus for instance for $k = \frac{1}{2}$ we obtain

$$r_m = [\tfrac{1}{2}(r_1^2 + r_2^2)]^{1/2}, \quad \text{if } J \ll J_0$$
$$= (r_1 r_2)^{1/2}, \quad \text{if } J \gg J_0/(2k).$$

Thus if the central conductor carries a return current which is a kth part of the discharge current, then for sufficiently high values of this current the discharge is localized in the chamber near $r = r_m$, this radius depending on k. The current remains detached from the walls and forms a thin-walled hollow cylinder.

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ON THE POSSIBILITY OF ACCELERATING POLARIZED PROTONS IN A CYCLOTRON*

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Abstract—The probability of spin re-orientation is estimated for polarized protons accelerated in a cyclotron in which the field falls off with increasing radius and varies azimuthally. Beam extraction is also considered. The depolarization probabilities are shown to be both small.

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1. INTRODUCTION

THE study of nuclear interaction processes and the explanation of the nature of nuclear forces demand experiments with polarized particles. At present polarized particles are obtained by scattering and therefore the beam intensity of such particles is small. Experiments involving the study of double or triple scattering of polarized particles are difficult to perform and are sometimes quite impossible owing to the smallness of the effects to be measured. In connexion with the development of a source of polarized ions[1] the question arose whether it is possible to accelerate and extract these ions from a cyclotron. If such acceleration is feasible the need for the first scattering is removed and, given a good source of polarized particles, one could obtain high intensity beams of these particles. We give below an approximate discussion of the acceleration and extraction of polarized particles from a cyclotron.

In generally accepted notation the Schroedinger equation for the spin function S has the form[2]

$$ih\frac{dS}{dt} = -(HM)\,S. \tag{1}$$

The spin function S is represented by a 2×2 matrix, so that equation (1) can be written in the form

$$\left.\begin{aligned}\frac{dS_1}{dt} &= \frac{i\mu H_z}{h}S_1 + \frac{i\mu}{h}(H_x - iH_y)\,S_2,\\ \frac{dS_2}{dt} &= \frac{i\mu}{h}(H_x + iH_y)\,S_1 - \frac{i\mu H_z}{h}S_2,\end{aligned}\right\} \tag{2}$$

where $\mu = \gamma eh/(2mc)$; (for a proton $\gamma \simeq 2{\cdot}8$).

If initially (at $t = 0$) the projection of the spin on the x-axis is positive ($S_1(0) = 1$; $S_2(0) = 0$), the probability of finding particles with spin of the opposite orientation at time t, i.e. the depolarization probability, is determined by the expression $P(t) = |S_2(t)|^2$.

If H_x and H_y vary harmonically,

$$\left.\begin{aligned}H_x &= H_1\cos\omega t,\\ H_y &= H_1\sin\omega t,\end{aligned}\right\} \tag{3}$$

solution of (2) gives the following formula for the depolarization probability*

$$P(t) = \frac{q^2\sin^2\vartheta}{1 + q^2 - 2q\cos\vartheta} \times \times \sin^2\left\{\frac{\omega}{2}(1 + q^2 - 2q\cos\vartheta)^{1/2}\,t\right\}, \tag{4}$$

where

$$\left.\begin{aligned}q^2 &= \left(\frac{\nu}{\omega}\right)^2(1 + 4\Delta^2),\\ \Delta &= H_1/2H_0,\\ \tan\vartheta &= H_1/H_0 = 2\Delta.\end{aligned}\right\} \tag{5}$$

The quantity ν entering in these expressions is the spin precession frequency in a field of intensity H_0:

$$\nu = 2\mu H_0/h. \tag{6}$$

Since the angular velocity of revolution of ions in a cyclotron is $\omega_c = eH_0/(mc)$, we have $\nu = \gamma\omega_c$. One usually has $\Delta = H_1/(ZH_0) \ll 1$ and $\tan\vartheta \approx \sin\vartheta \approx \vartheta \approx 2\Delta$, so that after some transformation equation (4) can be put in the form

$$P(t) = \frac{4\Delta^2}{[(\omega/\gamma\omega_c) - 1]^2 + 4\Delta^2} \times \times \sin^2\left\{\frac{\omega}{2}[(1 - \gamma\omega_c/\omega)^2 + 4\Delta^2(\gamma\omega_c/\omega)^2]^{1/2}\,t\right\}. \tag{7}$$

At resonance we have

$$\omega = \nu = \gamma\omega_c, \tag{8}$$

so that the probability amplitude for depolarization becomes unity and

$$P(t) = \sin^2(\omega\Delta t). \tag{9}$$

* Translated from *Atomnaya Energiya* **6**, 306 (1959).

* In BLOKHINTSEV's book the factor q^2 is omitted in the formula for $P(t)$.

2. DEPOLARIZATION OF PROTONS ON ACCELERATION

Using the above formulae we determine the depolarization probability for protons accelerated in a cyclotron whose magnetic field decreases with radius and varies azimuthally. The field components H_z, H_r and H_ϑ are not explicitly functions of time (in the absence of field fluctuations due to the power supply). They depend on time only through the co-ordinates r, ϑ and z, which vary as the particles are accelerated.

Let us put

$$H_z = -H_{00}\,[1 - \delta(r) + A(r)\cos k\vartheta], \tag{10}$$

where $\delta(r)$ takes into account the radial variation of the magnetic field while $A(r)$ describes the azimuthal inhomogeneity.

If the departure from the mean plane of the magnetic field is small we have

$$\left.\begin{aligned} H_r &= z\frac{\partial H_z}{\partial r} = zH_{00}\left(\frac{d\delta}{dr} - \frac{dA}{dr}\cos k\vartheta\right), \\ H_\vartheta &= \frac{z}{r}\frac{\partial H_z}{\partial \vartheta} = \frac{z}{r}H_{00}kA(r)\sin k\vartheta. \end{aligned}\right\} \tag{11}$$

Going over to Cartesian co-ordinates according to the formula

$$\left.\begin{aligned} H_x &= H_r\cos\vartheta - H_\vartheta\sin\vartheta, \\ H_y &= H_r\sin\vartheta + H_\vartheta\cos\vartheta \end{aligned}\right\} \tag{12}$$

and introducing the notation

$$\left.\begin{aligned} N &= \tfrac{1}{2}(kA/r + dA/dr), \\ M &= \tfrac{1}{2}(kA/r - dA/dr), \end{aligned}\right\} \tag{13}$$

we obtain

$$\left.\begin{aligned} H_x &= zH_{00}\left[\frac{d\delta}{dr}\cos k\vartheta - \right. \\ &\quad \left. - N\cos(k-1)\vartheta + M\cos(k+1)\vartheta\right], \\ H_y &= zH_{00}\left[\frac{d\delta}{dr}\sin k\vartheta + \right. \\ &\quad \left. + N\sin(k-1)\vartheta + M\sin(k+1)\vartheta\right]. \end{aligned}\right\} \tag{14}$$

Owing to the vertical oscillations of the particles in the cyclotron we have

$$z = z_m\cos\omega_v t, \tag{15}$$

where z_m is the amplitude of these oscillations and is a function of radius while ω_v is the frequency of vertical oscillations. For free oscillations in the decreasing magnetic field, we have

$$\omega_v = \sqrt{n}\,\omega_c, \tag{16}$$

where $n = -(r/H_z)(dH_z/dr)$ is the logarithmic index of field variation with radius. Since in a cyclotron we have $n < 1$ the quantity ω_v is always smaller than the frequency of revolution ω_c. One can take it that on the final radius we are in the range

$$n \approx 0{\cdot}1 - 0{\cdot}2; \quad \omega_v \approx (0{\cdot}3 - 0{\cdot}5)\omega_c.$$

In the region of small values of r equation (16) is inapplicable. Here the vertical oscillations are essentially determined by the electric field of the dees. In general one can put $\omega_v = p/\omega_c$ where $p \leqslant 0{\cdot}5$.

Taking into account the vertical oscillations and putting $\vartheta = \omega_c t$ the expression (14) may be written in the form

$$\left.\begin{aligned} H_x &= \tfrac{1}{2}z_m H_{00}\left\{\frac{d\delta}{dr}\left[\cos(1-p)\,\omega_c t\right.\right. \\ &\quad \left. + \cos(1+p)\,\omega_c t\right] \\ &\quad - N\left[\cos(k-p-1)\,\omega_c t\right. \\ &\quad \left. + \cos(k+p-1)\,\omega_c t\right] \\ &\quad + M\left[\cos(k-p+1)\,\omega_c t\right. \\ &\quad \left.\left. + \cos(k+p+1)\,\omega_c t\right]\right\}, \\ H_y &= \tfrac{1}{2}z_m H_{00}\left\{\frac{d\delta}{dr}\left[\sin(1-p)\,\omega_c t\right.\right. \\ &\quad \left. + \sin(1+p)\,\omega_c t\right] \\ &\quad + N\left[\sin(k-p-1)\,\omega_c t\right. \\ &\quad \left. + \sin(k+p-1)\,\omega_c t\right] \\ &\quad + M\left[\sin(k-p+1)\,\omega_c t\right. \\ &\quad \left.\left. + \sin(k+p+1)\,\omega_c t\right]\right\}. \end{aligned}\right\} \tag{17}$$

Spin reorientation can be caused only by a perturbing field whose vector rotates about the same direction as that around which the spin precesses. In the co-ordinate system we have chosen this is the case when the components H_x and H_y of the perturbing field are of the same sign. Consequently the terms in (17) containing the coefficient N may be neglected in the following discussion. In agreement with equation (8) resonance is reached for

$$1 \pm p = \gamma \tag{18}$$

or

$$k \pm p \pm 1 = \gamma. \tag{19}$$

In a cyclotron condition (18) is never fulfilled because $p < 1$ and for protons γ is 2·8. Let us estimate the depolarization probability for a proton, for instance for $p = 0{\cdot}4$. According to (5) we have for this case $\Delta = \frac{1}{4}z_m(d\delta/dr)$. Putting $\delta = b(r/R)^\beta$, $b = 0{\cdot}02$, $\beta = 10$, $z_m = 3$ cm and $r = R = 70$ cm (the final radius),

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we obtain $\Delta = 3 \times 10^{-3}$. The depolarization probability amplitude thus proves to be very small ($P \approx 5 \times 10^{-5}$).

The resonance conditions (19) may, however, be satisfied in a cyclotron. In that case $p = \gamma \pm k \pm 1$. Since k is an integer and p a positive quantity not greater than about 0·5 the resonance value of p can be attained only for $k = 2$, ($p = 0{\cdot}2$).

In ordinary cyclotrons the amplitude of the second harmonic of the azimuthal field variation ($k = 2$) is very small (~0·1 per cent of the main field), while in cyclotrons with azimuthal field variation one has $k > 2$. Nevertheless let us examine to what extent this resonance might be dangerous. According to formula (9) the maximum value of the probability of proton depolarization at resonance ($\omega = \gamma\omega_c$) is attained for

$$\omega \Delta t = \pi/2, \tag{20}$$

i.e. at the instant

$$t = T/4\gamma\Delta, \tag{21}$$

where T is the period of the particle revolution and is approximately equal to the period of the HF power on the dees. The amplitudes of the field components at resonance are determined from (17) and are of the form

$$H_1 = \tfrac{1}{2} z_m H_{00} M = \tfrac{1}{8} z_m H_{00}(A/r - dA/dr). \tag{22}$$

If $A = a(r/R)^\alpha$ we get

$$\Delta = H_1/(2H_0) = \tfrac{1}{8} a z_m \{(2/r)(r/R)^\alpha - (\alpha/r)(r/R)^{\alpha-1}\}.$$

For $\alpha = 2$ the quantity Δ proves to be zero and t becomes infinite. Thus when the particle passes through the dangerous resonance zone, there is no re-orientation of the spin. For another law of fall-off ($\alpha = 1$) we get $\Delta = a z_m/4R$. Taking $z_m = 3$ cm $R = 70$ cm and $a = 10^{-2}$ we obtain $\Delta \approx 10^{-4}$ and $t \approx 10^3\, T$. In cyclotrons the total number of periods of high frequency voltage during the acceleration of a particle is 50–100. Therefore for $\alpha = 1$ also there is no danger of a change of spin orientation. As the values of Δ obtained are small, it is evidently unnecessary to consider other laws of variation of $A(r)$.

If one takes into account terms of higher order in the expression for the components of magnetic field this should not lead to any noticeable increase in the probability of depolarization, in spite of the fact that new frequencies appear in equation (17). In fact the amplitudes of the additional harmonic terms will be very small in comparison with those already taken into account. This will ensure the smallness of Δ, the quantity which determines the permissible time for a particle to stay in the varying magnetic field. Therefore the permissible number of periods T in equation (21) increases and the probability of depolarization decreases.

If the source is mounted in the centre of the magnet this produces a displacement of the orbit centre which can be taken into account by introducing a first harmonic ($k = 1$) in the azimuthal inhomogeneity. The quantity $A(r)$, (see equation 10), will in this case be small as in the example examined above and therefore its influence on the depolarization may be neglected.

3. DEPOLARIZATION OF THE PROTONS ON EXTRACTION OF THE BEAM

When the beam is extracted the particle is moving in a strongly inhomogeneous magnetic field. Calculations and experiments have shown that in the deflecting field the ions perform about one vertical oscillation independently of the initial conditions (the distance from the median plane, z_0 and the initial phase at entry into the deflector). The ions pass through the median plane roughly in the region of maximum magnetic field gradient (see Fig. 1).

The radial component of the field H_r may be represented by the approximate expression

$$H_r(l) = z\frac{\partial H_z}{\partial r} = \frac{\partial H_z}{\partial r} z_0 \sin 2\pi l/l_f. \tag{23}$$

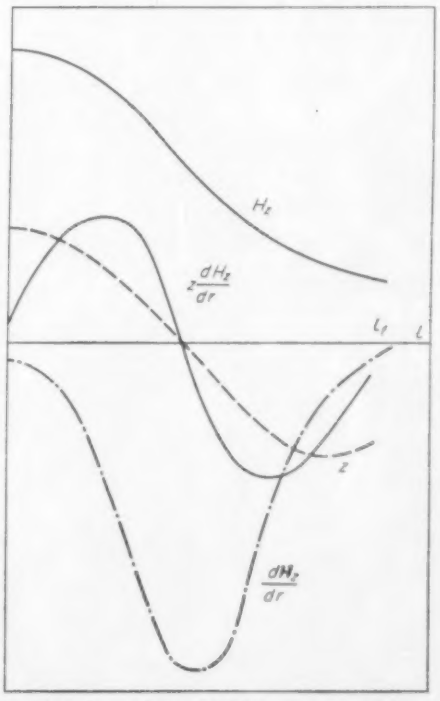

FIG. 1.—Variations of the field intensity H_z, of its radial derivative dH_z/dr, of the vertical position co-ordinate z and of $z(dH_z/dr)$ along the path l on the mean trajectory of an ion in the deflected beam. ($l = 0$ corresponds to entry into the deflector system.)

Since $l = vt$, we have $H_r(l) = z_0(\partial H_z/\partial r) \sin \kappa t$ where $\kappa = 2\pi v/l_f$.

During the time in which the particle traverses the path l_f from the entry into the deflector (radius r_i) to the final point of the path (exit from the chamber, radius r_f) the instantaneous value $\dot{\varphi}$ of the frequency varies within the limits

$$v/r_i \geqslant \dot{\varphi} \geqslant v/r_f.$$

The quantity v/r_i is equal to the frequency of revolution of a proton before its entry into the deflector (ω_c), while $r_f \simeq 2r_i$ so that the angular velocity $\dot{\varphi}$ at exit is between ω_c and $0{\cdot}5\ \omega_c$. For an estimate of the depolarization we use the field components

$$\left.\begin{aligned} H_x &= H_r \cos \dot{\varphi} t = z_0 \frac{\partial H_z}{\partial r} \sin \kappa t \cos \dot{\varphi} t, \\ H_y &= H_r \sin \dot{\varphi} t = z_0 \frac{\partial H_z}{\partial r} \sin \kappa t \sin \dot{\varphi} t. \end{aligned}\right\} \quad (24)$$

Therefore the expressions for the components of the perturbing field involve the frequencies

$$\Omega_{1,2} = \kappa \pm \dot{\varphi}, \quad (25)$$

in which the quantities κ and ω_c are comparable. The spin precession frequency $\nu = \gamma e H_z/(mc)$ varies during the process of extraction between the limits

$$\gamma\omega_c \geqslant \nu \geqslant \gamma(eH_z/mc)_{r=r_f}. \quad (26)$$

Comparing the relations (25) and (26) we may note that in the extraction of the particle there may be one or two points (if for $t = 0$ we have $\kappa + \omega_c < \gamma\omega_c$) in which the frequencies $\Omega_{1,2}$ are equal to the frequency of spin precession (resonance points).

For a very exaggerated estimate for the amount of depolarization we assume that the resonance takes place not at the points noted but in a broad region of values of l (from $l = \frac{1}{4}l_f$ to $l = \frac{3}{4}l_f$). The time of passage of the particles through this region is $t = l_f/(2v)$. We take the magnitude of the field intensity to be equal to some mean value $H_0 = \frac{1}{2}H_i$ (H_i is the field at $l = 0$) and the frequency Ω to have its maximum value $\Omega_{\max} = \kappa + \omega_c$. Therefore in agreement with expression (9) the probability of spin re-orientation proves to be equal to $P = \sin^2 \Omega_{\max} \Delta t$.

For the cyclotron of the Atomic Energy Institute of the U.S.S.R. Academy of Sciences one has for proton acceleration up to 12 MeV: $v = 0{\cdot}16c$, $H_i = 7{\cdot}6k$ Oe, $\omega_c = 75 \times 10^6$ sec^{-1}, $\kappa = 10^8$ sec^{-1}, $l_f \approx 300$ cm and $t = 3 \times 10^{-8}$ sec. The quantity $\Delta = z_0(\partial H_z/\partial r) \cdot \frac{1}{2}H_0$ for

$$z_0\, \partial H_z/\partial r \approx 200 \text{ Oe}$$

proves to be $\sim 0{\cdot}03$. For the chosen values of $\Omega_{\max}$ and t we have $P = 0{\cdot}03$. In actual fact t is considerably smaller than the value assumed earlier and the depolarization of protons on extraction from the cyclotron is many times smaller than these estimates would give.

To obtain polarized ions in the central region of the cyclotron many methods can be proposed. The choice of the method most effective in ensuring a beam of accelerated polarized ions of maximum intensity can be made only after a special investigation.

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NON-LINEAR THEORY OF SLOW PLASMA WAVES*

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Abstract—The propagation of electromagnetic waves in a narrow plasma waveguide is examined in a non-linear approximation. The dependence of the phase velocity on the amplitude is derived, and consideration is given to the problem of frequency multiplication. The amplitude of the second harmonic is calculated. Owing to non-linear effects a new possibility arises of varying the phase velocity of the wave by means of a variation of the amplitude and of ensuring radial and phase stability in accelerators and also of a certain modification of the methods of amplifying and generating micro-radio waves.

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1. INTRODUCTION

IT is well known that the dielectric constant of a plasma is $\varepsilon = 1 - (\Omega/\omega)^2$ and that therefore the phase velocity V_{ph} of waves propagated through unlimited plasma is greater than c, the velocity of light *in vacuo*. Only waves whose frequency is $\omega > \Omega$ can be propagated through a plasma because for $\omega < \Omega$ the dielectric constant ε of the plasma is negative.

A bounded plasma[1–3] has entirely different dispersion properties. In a plasma waveguide with cylindrical boundaries (a plasma rod or a channel inside a plasma) one can have wave propagation at frequencies $0 < \omega < \Omega/\sqrt{2}$, while the phase velocity of wave propagation V_{ph} may be less than c. The dispersion law and the topography of the field in a plasma waveguide are determined from a linear theory of wave propagation in such waveguides. However, with increasing field intensity and increasing wavelength the linear approximation becomes inapplicable. Therefore the question arises of studying non-linear propagation problems for micro-radio waves in plasma waveguides. The non-linear theory in the one-dimensional case has been studied by AKHIEZER and LYUBARSKY[3] and by SEN[5]. In the present paper we examine a particular two-dimensional case.

2. PROPAGATION OF PLASMA WAVES IN THE HYDRODYNAMIC APPROXIMATION

Since strong electromagnetic fields are of greatest interest we shall start from the hydrodynamic approximation, i.e. we shall neglect the temperature of the plasma ($T = 0$) as well as the distribution function. The waveguide properties of plasma show themselves even at small densities. Collisions in the plasma may therefore be neglected.

We consider the propagation of electromagnetic waves in a plasma in the case when the wave propagation is not associated with a variable component of the density ($\rho_\sim = 0$). Then the system of equations describing the plasma wave propagation is

$$\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\rho \mathbf{V}, \tag{1}$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\frac{e}{m}\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{2}$$

Here $\mathbf{V}$ is the velocity of the plasma electrons. It is assumed that the ions are stationary. Since $\rho_\sim = 0$ we assume that $\varphi = 0$. It then follows from the Lorentz condition that

$$\operatorname{div} \mathbf{A} = 0. \tag{3}$$

The equation of continuity then has the form

$$\operatorname{div} \mathbf{V} = 0. \tag{4}$$

We shall seek to solve the equations for all the quantities in equations (1)–(4) in the form $R(r)f(z - Vt)$, where z is the direction of wave propagation and V the phase velocity.

Using equation (1) we obtain for the component A_z the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{\partial^2 A_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} = -\frac{4\pi}{c}\rho_0 V_z, \tag{5}$$

where ρ_0 is the equilibrium electron density.

We introduce the dimensionless quantities

$$\mathbf{A}_0 = \frac{\mathbf{A}}{\frac{mc}{e}V}, \quad r_0 = r\frac{\Omega}{V}, \quad z_0 = z\frac{\Omega}{V},$$

$$\tau = \Omega t, \quad V_{0z} = \frac{V_z}{V},$$

where $\Omega^2 = 4\pi e^2 n/m$ is the square of the Langmuir

* Translated from *Atomnaya Energiya* **6**, 447 (1959)

frequency of plasma electron oscillations. We rewrite equation (5) in the form

$$\frac{1}{r_0}\frac{\partial}{\partial r_0}\left(r_0\frac{\partial A_{0z}}{\partial r_0}\right)+\frac{\partial^2 A_{0z}}{\partial z_0^2}-\beta^2\frac{\partial^2 A_{0z}}{\partial \tau^2}=-\beta^2 V_{0z}, \quad (6)$$

and

$$\beta = V/c$$

We use equations (2) and (4) and obtain

$$\frac{\partial V_{0z}}{\partial \tau}-\frac{r_0}{2}\frac{\partial V_{0z}}{\partial z_0}\frac{\partial V_{0z}}{\partial r_0}+V_{0z}\frac{\partial V_{0z}}{\partial z_0}=-\frac{\partial A_{0z}}{\partial \tau}.$$

Passing from the variable $\eta = z - Vt = (V/\Omega)(z_0 - \tau)$, which has the dimensions of length, to the dimensionless variable $\eta_0 = z_0 - \tau = \eta\Omega/V$ we seek to determine all quantities as functions of the form $R(r_0)f(z_0 - \tau)$. We write equations (5) and (6) as

$$\frac{1}{r_0}\frac{\partial}{\partial r_0}\left(r_0\frac{\partial R}{\partial r_0}\right)f(\eta_0)+R(r_0)(1-\beta^2)\frac{\partial^2 f}{\partial \eta_0^2} = -\beta_0^2 V_{0z}(r_0)V_{0z}(\eta_0), \quad (7)$$

$$\frac{\partial V_{0z}}{\partial \eta_0}\left[V_{0z}-1-\frac{r_0}{2}\frac{\partial V_{0z}}{\partial r_0}\right]=\frac{\partial A_{0z}}{\partial \eta_0}. \quad (8)$$

3. PROPAGATION IN THEIR PLASMA WAVEGUIDES

It is an extremely difficult mathematical problem to solve the system of equations (7) and (8) in a general form. In the following we shall consider thin rod-like plasma waveguides, i.e. we shall assume that $r_0^2 \ll 1$.

Although such an assumption restricts the region of applicability of the solutions to be obtained it allows one to clarify the basic question of the difference in the propagation of electromagnetic waves in a limited and an unlimited plasma. In this sense the assumption that r_0 is small is the most appropriate because with it the properties of the bounded plasma must be most pronounced.

Since the radius of the plasma rod is assumed to be small we can restrict ourselves to the first terms of the power series expansions of $R(r_0)$ and $V_{0z}(r_0)$.

$$R(r_0) = R(0) + R'(0)r_0 + \frac{R''(0)r_0^2}{2},$$

$$V_{0z}(r_0) = V_{0z}(0) + V'_{0z}(0)r_0 + \frac{V''_{0z}(0)r_0^2}{2}.$$

Then

$$\frac{1}{r_0}\frac{\partial}{\partial r_0}\left(r_0\frac{\partial R}{\partial r_0}\right)=\frac{1}{r_0}R'(0)+2R''(0).$$

For $r_0 = 0$ the quantity V_{0z} is finite. It therefore follows from equation (7) that $R'(0) = 0$. According to equation (7) the dependence of $V_{0z}(r_0)$ has the same form as that of $R(r_0)$ so that

$$\frac{1}{r_0}\cdot\frac{\partial}{\partial r_0}\left(r_0\frac{\partial R}{\partial r_0}\right)=2R''(0)=\text{const}.$$

Thus

$$\left.\begin{aligned} R(r_0) &= R(0)+\frac{R''(0)r_0^2}{2},\\ V_{0z}(r_0) &= V_{0z}(0)+\frac{V''_{0z}(0)r^2}{2}. \end{aligned}\right\} \quad (9)$$

Keeping in equation (7) terms of zero order in r_0, we obtain

$$2R''(0)f(\eta_0)+R(0)\frac{d^2 f}{d\eta_0^2}(1-\beta^2) = -\beta^2 V_{0z}(0)V_{0z}(\eta_0),$$

or

$$\frac{d^2 f}{d\eta_0^2}(1-\beta^2)+k_1^2 f = -\beta^2 V'_{0z}(0)V_{0z}(\eta_0), \quad (10)$$

where

$$k_1^2 = \frac{2R''(0)}{R(0)}.$$

If in particular $R(r_0) \approx J_0(r_0)$ i.e. if the dependence on r_0 is the same as in the linear approximation, we obtain, taking into account that

$$\frac{1}{r_0}\frac{\partial}{\partial r_0}\left(r_0\frac{\partial R}{\partial r_0}\right)=k_1^2 R,$$

the relation

$$\left[\frac{d^2 f}{d\eta_0^2}(1-\beta^2)+k_1^2 f\right]\alpha J_0(r_0) = -\beta^2 V_{0z}(r_0)V_{0z}(\eta_0). \quad (11)$$

Hence it follows that $V_{0z}(r_0) \approx J_0(r_0)$. Using the expressions (8) and (9) we obtain

$$V_{0z}-1-\frac{r_0}{2}\frac{\partial V_{0z}}{\partial r_0} = V_{0z}(0)-1+O(r_0^2),$$

$$V_{0z}(0)\frac{dV_{0z}(\eta_0)}{d\eta_0}[V_{0z}(\eta_0)V_{0z}(0)-1] + O(r_0^2) = \frac{df}{d\eta_0}R(0). \quad (12)$$

Thus for thin strands of plasma the equation of motion up to terms of order r_0^2 has the same form as for a homogeneous motion.

Integrating equation (12) we obtain

$$V_{0z}(\eta_0) = 1-\sqrt{C_1+2fR'(0)}. \quad (13)$$

Using the expressions (10) and (13) we obtain the following equation for determining f:

$$\frac{d^2 f}{d\eta_0^2}(1-\beta^2)+k_1^2 f = -\beta^2[1-\sqrt{C_1+\alpha f}] \quad (14)$$

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The solution of equation (14) may be obtained in a general form in terms of elliptic integrals.

We consider first slow waves ($\varepsilon = (V/c)^2 \ll 1$). Then equation (14) takes on the form

$$\frac{d^2f}{d\eta^2} + k_1^2 f = -\varepsilon[k_1^2 f + (1 - \sqrt{C_1 + 2f})] + \varepsilon^2\left[\frac{k_1^2 f}{2} - (1 - \sqrt{C_1 + 2f})\right]. \quad (15)$$

In order to solve equation (15) we use the method for solving non-linear equations with a small non-linear term worked out by BOGOLYUBOV. We seek the solution in the form

$$f = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \ldots, \quad (16)$$

where the u_1 are periodic functions of period 2π while the quantities a and Ψ are determined by the equations

$$\left.\begin{aligned} \frac{da}{d\eta} &= \varepsilon_1 A_1(a) + \varepsilon^2 A_2(a), \\ \frac{d\psi}{d\eta} &= k_1 + \varepsilon B_1(a) + \varepsilon^2 B_2(a). \end{aligned}\right\} \quad (17)$$

To make our definitions of u_i, A_i and B_i unambiguous we subject the functions u_i to the conditions

$$\int_0^{2\pi} u_i(a,\psi)\cos\psi\, d\psi = 0, \quad \int_0^{2\pi} u_i(a,\psi)\sin\psi\, d\psi = 0.$$

This means that the quantity a is the total amplitude of the first harmonic in the oscillation.

Inserting the function (16) into (15) and using equation (17) we obtain

$$\begin{aligned} \frac{d^2f}{d\eta^2} + k_1^2 f = \varepsilon &\left\{-2k_1A_1\sin\psi - 2k_1aB_1\cos\psi + k_1^2\frac{\partial^2 u_1}{\partial\psi^2} + k_1^2 u_1\right\} \\ &+ \varepsilon^2\left\{\left(A_1\frac{dA_1}{da} - aB_1^2 + 2k_1aB_2\right)\cos\psi - 2\left(k_1A_2 + 2A_1B_1 + A_1\frac{dB_1}{da}a\right)\sin\psi \right. \\ &+ 2k_1A_1\frac{\partial^2 u_1}{\partial\psi^2} \\ &\left. + 2k_1B_1\frac{\partial^2 u_1}{\partial\psi^2} + k_1^2\frac{\partial^2 u_2}{\partial\psi^2} + k^2 u_1\right\} + \ldots \quad (18) \end{aligned}$$

The right-hand side of equation (15) has the form $\varepsilon F_1(f) + \varepsilon^2 F_2(f)$. Therefore keeping in equation (16) terms up to ε^2 inclusive we obtain

$$\begin{aligned} &\varepsilon F_1(f) + \varepsilon^2 F_2(f) = \varepsilon F_1(a\cos\psi) \\ &+ \varepsilon^2[F_2(a\cos\psi) + u_1(a,\psi) + F_1'f(a\cos\psi)]. \quad (19) \end{aligned}$$

Equating terms of the same order in ε and performing calculations analogous to those performed by BOGOLYUBOV and MITROPOL'SKY[6] we find

$$u_1(a,\psi) = \frac{q_0(a)}{k_1^2} + \frac{1}{k_1^2}\sum_{n=1}^{\infty}\frac{q_n\cos n\psi}{1-n^2} \quad (20)$$

where the q_n are the expansion coefficients in the Fourier series for $F_1(f)$:

$$F_1(f) = q_0 + \sum_{n=1}^{\infty} q_n(a)\cos n\psi.$$

Owing to the even character of the function F_1 the coefficients of $\sin n\psi$ are zero. The quantities A_1 and B_1 in equation (17) are determined by the relations

$$q_1 + 2k_1aB_1 = 0, \quad A_1 = 0. \quad (21)$$

The relation $A_1 = 0$ is explained by the absence of terms in $\sin\psi$ in the expansion of the function $F_1(f)$ (F_1 depends only on $\cos\psi$ owing to the conservative character of the system). The relations (20) and (21) determine A_1, B_1 and $u_1(a,\psi)$. Inserting the values of A_1 and B_1 into (17) one can find a and ψ accurate to terms proportional to ε.

To determine A_2, B_2 and $u_2(a,\psi)$ one must compare coefficients of terms proportional to ε^2, on the right hand sides of equations (18) and (19) and one must use the relations (20) and (12) determining A_1, B_1 and $u_1(a,\psi)$. Performing the necessary calculations we get

$$\left.\begin{aligned} &q_1{}^1 + 2k_1aB_2 = 0, \quad A_2 = 0, \\ &u_2(a,\psi) = q_0{}^{(1)} + \frac{1}{k_1^2}\sum_{n=2}^{\infty}\frac{q_n{}^{(1)}\cos n\psi}{1-n^2}, \end{aligned}\right\} \quad (22)$$

where

$$\begin{aligned} q_1{}^{(1)} &= \frac{1}{\pi}\int_0^{2\pi} F_2(a\cos\psi)\, d\psi \\ &\quad + \frac{1}{\pi}\int_0^{2\pi} u_1(F_1'f + 2B_1k_1)\cos\psi\, d\psi + \frac{4q_1^2}{k_1^2a^2} \\ &= \frac{1}{\pi}\int_0^{2\pi} F_2(a\cos\psi)\, d\psi + \frac{1}{k_1^2}\sum_{n=2}^{\infty}\frac{q_n}{1-n^2}\frac{dq_n}{da} \\ &\quad + \frac{2}{k_1^2}q_0\frac{dq_0}{da} + \frac{4q_1^2}{k_1^2a}, \end{aligned}$$

$$\begin{aligned} q_n{}^{(1)} &= \frac{1}{\pi}\int_0^{2\pi} F_2(a\cos\psi)\cos n\psi\, d\psi \\ &\quad + \frac{1}{\pi}\int_0^{2\pi} u_1(a,\psi)F_i'f\cos n\psi + \frac{4}{k_1^2a}q_1q_n. \end{aligned}$$

These relations determine the function $u_2(a,\psi)$. To find the amplitude of the first harmonic in a and the phase ψ up to terms of second order one must insert A_2 and B_2 into (17) and then solve this equation.

According to (22) we have $B_2 = -q_1^{(1)}/(2k_1a)$. Confining oneself to the first term of the sum appearing in the expression q_1 we obtain

$$q_1^{(1)} = \frac{1}{\pi}\int_0^{2\pi} F_2(a\cos\psi)\,d\psi - \frac{1}{3}q_2\frac{dq_2}{da} + \frac{2}{k_1^2}q_0\frac{dq_0}{da} + \frac{4q_1^2}{k_1^2 a}.$$

Taking into account the fact that $F_2 = k_1^2 f/2 - (1-\sqrt{(C_1+2f)})$ we obtain

$$q_1^{(1)} = \frac{1}{\pi}\left[-2\pi + \sqrt{C_1}\,E(k)\right] - \frac{1}{3}q_2\frac{dq_2}{da} + \frac{2q_0}{k_1^2}\frac{dq_0}{da} + \frac{4q_1^2}{k_1^2 a}.$$

The quantities q_2, q_1 and q_0 have been defined above. Let us now solve equation (15) accurately to terms of order ε. To do this one must insert into (17) the quantities A_1 and B_1 from (21). Taking into account that in the case considered

$$F_1(f) = -k_1^2 f - (1-\sqrt{C_1+2f}),$$

we obtain for a and ψ the equations

$$\frac{da}{d\eta} = 0, \tag{23}$$

$$\left(\frac{d\psi}{d\eta}\right)^2 = k_1^2 + \frac{\varepsilon}{a}\frac{1}{\pi}\left\{k_1^2 a\pi + \int_0^{2\pi}\sqrt{C_1+2a\cos\psi}\,\cos\psi\,d\psi\right\}. \tag{24}$$

It follows from (23) that up to terms of first order in ε inclusive the amplitude of the first harmonic is independent of η. Owing to the conservative nature of the system considered this conclusion is valid including terms of any order in ε. Thus $a = \text{const}$.

One can show that after a series of transformations and calculations the integral on the right-hand side of equation (24) is equal to

$$I = \frac{4}{3}\frac{\sqrt{C_1}}{\gamma a}\sqrt{1+\gamma a}\,[(\gamma a - 1)K(k) - E(k)]. \tag{25}$$

Here $k^2 = 2\gamma a/(1+\gamma a)$, $\gamma = 2a/C_1$ and $K(k)$ and $E(k)$ are respectively the complete elliptic integrals of the first and second kind. Thus we obtain for ψ the equation

$$\left(\frac{d\psi}{d\eta}\right)^2 = k_1^2\left(1+\frac{V^2}{c^2}\right) - \frac{4}{3}\frac{V^2}{c^2}\frac{\sqrt{C_1}}{\pi} \times [(\gamma a - 1)K(k) - E(k)]\frac{\sqrt{1+\gamma a}}{\gamma a}. \tag{26}$$

Equation (26) involves the arbitrary constant C_1 which may be determined by the following considerations. The velocity of a plasma particle can be found in general from the relation $V_z = 1 - \sqrt{(C_1+2f)}$. If the plasma as a whole is at rest the mean velocity (averaged over an oscillation period of η) is zero, i.e.

$$\bar{V}_z = \int_0^{2\pi} V_z\,d\psi = 2\pi - \sqrt{C_1}\int_0^{2\pi}\sqrt{1+\gamma f}\,d\psi = 0. \tag{26'}$$

Using the first approximation for f we obtain

$$\sqrt{C_1} = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{1+\gamma a\cos\psi}\,d\psi.$$

Since

$$\int_0^{2\pi}\sqrt{1+\gamma a\cos\psi}\,d\psi = 4(1+\gamma a)^{1/2}E(k),$$

we have

$$\sqrt{C_1} = \frac{\pi}{2}\frac{1}{E(k)(1+\gamma a)^{1/2}}. \tag{27}$$

For a moving plasma strand $\bar{V}_z = V_0$, the velocity of the strand while the constant C_1 is determined by the condition

$$\sqrt{C_1} = \frac{\pi}{2}\frac{1}{E(k)(1+\gamma a)^{1/2}}\left(1-\frac{V_0}{V}\right).$$

4. THE DISPERSION LAW IN THIN PLASMA WAVEGUIDES

We consider now the particular case of a sinusoidal wave. Then $\psi = \omega t - k_3 z = -\eta\omega/\Omega$ and equation (26) takes on the form

$$\left(\frac{\omega}{\Omega}\right)^2 = k_1^2\left(1+\frac{V^2}{c^2}\right) - \frac{4}{3}\frac{\sqrt{1+\gamma a}}{\gamma a}\frac{\sqrt{C_1}}{\pi} \times [(\gamma a - 1)K(k) - E(k)]. \tag{28}$$

This is the dispersion law for the problem we are considering. In the non-linear treatment the phase velocity of the wave propagation depends not only on frequency but also on the amplitude a of the waves.

Let us examine equation (28) in more detail for $\gamma a \ll 1$. We then have $k \ll 1$. Using the series expansions for $K(k)$ and $E(k)$ for small values of the variables we obtain

$$\left(\frac{\omega}{\Omega}\right)^2 = k_1^2\left(1+\frac{V^2}{c^2}\right) - \frac{V^2}{c^2}\frac{\sqrt{C_1}}{2}(1+\gamma^2 a^2). \tag{29}$$

The constant determined by equation (27) is $C_1 = 1 + a^2/8$. Thus the dispersion law (29) takes on the form

$$\left(\frac{\omega}{\Omega}\right)^2 = k_1^2\left(1+\frac{V^2}{c^2}\right) - \frac{V^2}{c^2} - \frac{3}{8}\frac{V^2}{c^2}a^2.$$

For $a \to 0$ it goes over into the linear dispersion law

$$k_1^2 = \frac{\omega^2}{V^2} - \frac{\omega^2}{c^2}\left(1-\frac{\Omega^2}{\omega^2}\right),$$

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where k_1 is determined by equation (1) of the papers by SCHUMANN[1] or FAINBERG.[2]

As one would expect, the non-linearity has led to a dependence of phase velocity on amplitude. Another consequence of the non-linearity is the appearance of harmonics, i.e. a deviation of the wave form from the sinusoidal. This deviation is determined to first order in ε by the function $u_1(a,\psi)$ which allows one to find the true form of the wave.

Let us determine the amplitude of the second harmonic. (One can find higher harmonics by similar methods.) According to relation (20) the amplitude of the second harmonic is $a_2 = -q_2/3k_1^2$ where

$$q_2 = \frac{1}{\pi}\sqrt{C_1}\int_0^{2\pi}\sqrt{1+\gamma a\cos\psi}\cos 2\psi\, d\psi.$$

After a certain amount of calculation we obtain

$$\int_0^{2\pi}\sqrt{1+\gamma a\cos\psi}\cos 2\psi\, d\psi = \frac{4}{15}\frac{\sqrt{1+\gamma a}}{(\gamma a)^2}\{4(1-\gamma a)K(k) - (4-3\gamma^2a^2)E(k)\}.$$

Therefore

$$a_2 = -\frac{1}{3k_1^2}\frac{4}{15\pi}\sqrt{C_1}\frac{\sqrt{1+\gamma a}}{(\gamma a)^2}\times \times\{4(1-\gamma a)K(k) - (4-3\gamma^2a^2)E(k)\}. \quad (30)$$

If $\gamma a \ll 1$ we find, using the expressions (27) and (30),

$$a_2 = -\frac{64}{15\pi}\frac{a^2}{(1+2a)^{5/2}}. \quad (31)$$

With the aid of equation (26) and the relation (20) the problem of the propagation of electromagnetic waves in plasma waveguides is completely solved to the order ε.

In solving equation (14) the quantity $\varepsilon = \beta^2 = (V/c)^2$ was assumed to be small. Let us now solve this equation without the assumption that β^2 is small, but assuming instead that $2f/C_1 \ll 1$. We re-write equation (14) in the form

$$\frac{d^2z}{d\eta^2} + \frac{k_1^2}{1-\beta^2}z = -\frac{2\beta^2}{1-\beta^2}\left[1-\sqrt{C_1}\sqrt{1+\frac{z}{C_1}}\right].$$

Assuming that $z/C_1 \ll 1$, performing a series expansion and dropping terms higher than of third order, we obtain

$$\frac{d^2z}{d\eta^2} + \omega^2 z = -\frac{2\beta^2}{1-\beta^2}(1-\sqrt{C_1}) + \frac{\beta^2}{4(1-\beta^2)}\left(-\frac{z^2}{C_1^2}+\frac{z^3}{2C_1^3}\right). \quad (32)$$

Assuming the quantity

$$\frac{\beta^2}{4(1-\beta^2)}\left(-\frac{z^2}{C_1^2}+\frac{z^3}{2C_1^3}\right) = \varepsilon f(z)$$

to be small we seek the solution of (32) up to terms in ε. Then

$$z = a\cos\psi,$$

$$\frac{da}{d\eta} = \varepsilon A_1, \quad \frac{d\psi}{d\eta} = \omega + \varepsilon B_1.$$

We determine the constant C_1 from the condition (26′). In our case this condition has the form

$$\sqrt{C_1}\int_0^{2\pi}\sqrt{1+\frac{a}{C_1}\cos\psi}\, d\psi = 2\pi.$$

Since a/C_1 is assumed small we have $C_1 = 1 + a^2/8$. Therefore equation (32) assumes the form

$$\frac{d^2z}{d\eta^2} + \omega^2 z = \frac{\beta^2}{4(1-\beta^2)}\left(\frac{z^3}{2}-z^2\right) + O(a^2). \quad (33)$$

Using (21) we obtain

$$\frac{da}{d\eta} = 0, \quad a = \text{const},$$

$$\left(\frac{d\psi}{d\eta}\right)^2 = \omega^2 - \frac{1}{8a\pi}\frac{\beta^2}{1-\beta^2}\times \times\int_0^{2\pi}\left[\frac{a^3\cos^3\psi}{2} - a^2\cos^2\psi\right]\cos\psi\, d\psi.$$

Taking into account that $z = 2f$ we obtain finally

$$\left(\frac{d\psi}{d\eta}\right)^2 = \frac{k_1^2}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} - \frac{3}{8}\frac{a^2\beta^2}{1-\beta^2}. \quad (34)$$

For a sinusoidal wave we have

$$\left(\frac{\omega}{\Omega}\right)^2 = \frac{k^2}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} - \frac{3}{8}\frac{a^2\beta^2}{1-\beta^2}. \quad (35)$$

For small β this dispersion law goes over into equation (30).

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LONGITUDINAL PLASMA OSCILLATIONS*

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Abstract—The present paper is a coherent account of various aspects of longitudinal oscillations in one and two component plasmas. A discussion is offered of dispersion equations, conditions necessary for the growth or decay of oscillations, the physical mechanisms of growing or damping, and the possibility of arbitrary steady-state solutions. The physical situation is described in terms of Poisson's equation and the Boltzmann equation, while the mathematical description is in terms of solutions of an initial-value problem in the small amplitude (linearized) approximation. Some general results are derived for an arbitrary unperturbed velocity distribution of electrons and ions. From these expressions the customary results for a stationary plasma in thermal equilibrium can readily be obtained. For simplicity, one dimensional motion of a simple one component plasma is treated in detail; appropriate generalizations for two or more component plasmas (electrons and ions) are, however, indicated in text. Collisions between particles and non-linear effects are not considered, nor are the effects of external electric or magnetic fields.

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1. INTRODUCTION

THE possibility of oscillation in a plasma due to local separation of charge and the consequent restoring forces has been known for a long time. Various aspects of the phenomenon have been treated by different workers, and the essential characteristics established (TONKS and LANGMUIR, 1929; VLASOV, 1945; LANDAU, 1946; BOHM and GROSS, 1949). The basic frequency of oscillation for a simple plasma in which only one species of particle (charge e, mass m) oscillates is

$$\omega_p^2 = \frac{4\pi e^2}{m} n_0 \tag{1.1}$$

where n_0 is the number of particles per unit volume. There is actually a slight dependence of the frequency on wave number, but for long wavelengths (large compared to the Debye length) the frequency is practically given by (1.1).

The fact that plasma oscillations can, under certain circumstances, be damped even in the absence of collisions was first explored by LANDAU (1946). He showed that an initial small disturbance in an electronic plasma in thermal equilibrium oscillates with the plasma frequency (1.1) but is damped with a damping constant:

$$\mathrm{Im}\,(\omega) \simeq -\omega_p \sqrt{\frac{\pi}{8}} \left(\frac{\omega_p}{kc}\right)^3 \exp\left(\frac{-\omega_p^2}{2k^2c^2}\right) \tag{1.2}$$

where k is the wave number of the disturbance and c is the root mean square thermal velocity. The damping is small for small wave numbers, but becomes appreciable for wave numbers of order the Debye wave number, $k_D = \omega_p/c$. This implies that the organized motion of plasma oscillations cannot be sustained for wavelengths shorter than the Debye length, a conclusion that can be obtained on other grounds. BOHM and GROSS (1949) point out that the physical mechanism of the damping described by (1.2) is the trapping by the electric field of particles moving at approximately the phase velocity of the wave, with a consequent exchange of energy between particles and plasma oscillations.

VAN KAMPEN (1955) and BERNSTEIN, GREENE and KRUSKAL (1957) show that, in contrast to the seemingly general results (1.1) and (1.2), it is possible to have steady-state plasma oscillations of *arbitrary* wave number and *arbitrary* frequency. Indeed, large amplitude oscillations of arbitrary wave form can be created (BERNSTEIN *et al.*, 1957). The difference between these arbitrary solutions of the plasma equations and the standard results (1.1) and (1.2) lies in the initial preparation of the plasma. A sufficiently clever distribution of particles trapped in the potential wave and of untrapped particles allows the creation of an arbitrary steady-state solution. The customary behaviour of a plasma, according to (1.1) and (1.2), results if the initial disturbance is chosen in a less careful way.

Recently, BUNEMAN (1958) has pointed out that relative motion of the electrons and ions in a plasma can give rise to growing plasma oscillations, and that

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these growing waves provide a mechanism for loss of translational energy by the particles. In the presence of an applied electric field the accelerated motion of electrons relative to ions will, in principle, be stopped by this generation of growing plasma waves. Such an energy loss provides a mechanism for an electrical resistivity of a plasma, even in the complete absence of collisions. Presumably this mechanism will govern the resistivity of a plasma at high temperatures where the collision resistivity falls to very small values.*

The present paper attempts to give a coherent account of the various aspects of longitudinal plasma oscillations, including a discussion of dispersion equations, conditions necessary for the growth or decay of oscillations, the physical mechanism of growing or damping and the possibility of arbitrary steady-state solutions. The mathematical description is in terms of an initial-value problem in the small amplitude approximation. Some general results are derived from an arbitrary unperturbed velocity distribution of electrons and ions. From these expressions the customary results for a stationary plasma in thermal equilibrium can be readily obtained. Collisions between particles are ignored. For a discussion of the effects of collisions on plasma oscillations the reader may consult the papers of BHATNAGAR, GROSS, and KROOK (1954) and GROSS and KROOK (1956). Non-linear effects are not treated, except for mention of the steady-state solutions of BERNSTEIN, GREENE, and KRUSKAL (1957).

2. ELEMENTARY DISCUSSION

As an introduction to the ideas involved a simple, nonrigorous derivation of the plasma frequency will be given. The equations governing a simple plasma are the Boltzmann equation:†

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f}{\partial v} = 0 \tag{2.1}$$

and Poisson's equation for the average electric field E:

$$\frac{\partial E}{\partial x} = 4\pi e(n_0 - \int f\, dv) \tag{2.2}$$

where $f(v, x, t)$ is the distribution function of the particles (charge $-e$, mass m) which can move relative to a stationary background of positive charge having charge density en_0. One-dimensional motion has been assumed for simplicity and collisions are ignored.

The simplest description of a plasma is in terms of the hydrodynamic approximation. The spread in velocities due to thermal motion is largely ignored, and the plasma is described by the hydrodynamic variables, particle density, average velocity and perhaps pressure, all assumed to be single-valued functions of space and time. The hydrodynamic equations can be written down directly or obtained by taking the first few velocity moments of the Boltzmann equation (2.1). For a one-component, one-dimensional plasma, let $n(x, t)$, $v(x, t)$, $p(x, t)$ be the particle density, mean velocity and pressure respectively. Then the zeroth and first moment equations are the continuity equation:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \tag{2.3}$$

and the equation of motion (for particles of charge $-e$):

$$m\frac{dv}{dt} + \frac{1}{n}\frac{\partial p}{\partial x} + eE = 0, \tag{2.4}$$

while Poisson's equation is:

$$\frac{\partial E}{\partial x} + 4\pi e(n - n_0) = 0. \tag{2.5}$$

If the particle density and velocity have small fluctuations n_1 and v_1 around their equilibrium values of n_0 and zero respectively, the hydrodynamic equations can be linearized to yield:

$$\left.\begin{aligned} \frac{\partial n_1}{\partial t} + n_0\frac{\partial v_1}{\partial x} &= 0 \\ \frac{\partial v_1}{\partial t} + \frac{e}{m}E &= 0 \\ \frac{\partial E}{\partial x} + 4\pi e n_1 &= 0 \end{aligned}\right\} \tag{2.6}$$

where we have dropped the pressure term from the force equation. Differentiation of the first equation with respect to time and substitution from successive equations gives an equation of motion for the density fluctuations:

$$\frac{\partial^2 n_1}{\partial t^2} + \frac{4\pi n_0 e^2}{m} n_1 = 0 \tag{2.7}$$

This shows that in the simple hydrodynamic approximation the plasma can undergo oscillations with the plasma frequency (1.1) (TONKS and LANGMUIR, 1929).

If the pressure gradient term in (2.4) had been retained in (2.6) and an equation of state specified, (2.7) would have contained a term in $\frac{\partial^2 n_1}{\partial x^2}$ which would give a dependence of frequency on wave number (i.e. a dispersion relation). Since one of our main concerns will be the effects of finite thermal velocity spreads, we will not pursue the hydrodynamic approximation further, but will return to the collisionless Boltzmann equation and discuss the dispersion relation in terms of it.

Starting with (2.1) and (2.2) small amplitude (linearized) equations are obtained by assuming that

* A crude argument shows that the resistivity due to generation of plasma oscillations should be roughly proportional to the reciprocal of the plasma frequency (1.1) and independent of temperature; whereas the collision resistivity (SPITZER L. (1956) *Physics of Fully Ionized Gases*, p. 83, Interscience, New York) is independent of density and varies with temperature approximately as $T^{-3/2}$.

† For a derivation of the collisionless Boltzmann equation as a first approximation to the rigorous system of equations describing the motion of a plasma, see KADOMTSEV (1957).

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the distribution function can be written $f = f_0 + f_1$, where $f_0 = f_0(v)$ is some unperturbed velocity distribution corresponding to n_0 particles per unit volume and $f_1 = f_1(v, x, t)$ is a small modification in the distribution. In the absence of external fields the electric field E is proportional to f_1 and so the linearized equations of motion become:

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{eE}{m}\frac{\partial f_0}{\partial v} = 0 \tag{2.8}$$

and

$$\frac{\partial E}{\partial x} = -4\pi e \int f_1\, dv. \tag{2.9}$$

With the assumption that f_1 and E are proportional to $\exp(ikx - i\omega t)$ equation (2.8) can be solved for f_1 in terms of E:

$$f_1 = \frac{1}{i(kv - \omega)}\frac{eE}{m}\frac{\partial f_0}{\partial v}. \tag{2.10}$$

This can be substituted into (2.9) to yield the relation:

$$\frac{k^2}{\omega_p{}^2} - \int \frac{G(v)\, dv}{v - \dfrac{\omega}{k}} = 0 \tag{2.11}$$

where $\omega_p{}^2$ is given by (1.1) and the function $G(v)$ is derivative of the normalized velocity distribution:

$$G(v) = \frac{1}{n_0}\frac{\partial f_0}{\partial v}. \tag{2.12}$$

The relation (2.11) is called a *dispersion relation* because in principle it yields the frequency ω as a function of k for any given velocity distribution $f_0(v)$. There is a difficulty, however, in that for real ω and k the integral is undefined because of the singularity in the integrand. If in an arbitrary manner we interpret the integral in a principal value sense, there will result a well defined relation $\omega = \omega(k)$. As an illustration consider that $f_0(v)$ is a Maxwellian distribution, or something like it, so that the integrand in (2.11) falls off rapidly for large positive or negative velocities. Furthermore, consider a small enough wave number that ω/k is large compared to the dominant velocities in $G(v)$. Then the denominator can be expanded and (2.11) becomes

$$\frac{k^2}{\omega_p{}^2} \simeq -\frac{k}{\omega}\int G(v)\left(1 + \frac{vk}{\omega} + \frac{v^2k^2}{\omega^2} + \cdots\right) dv. \tag{2.13}$$

Using the definition (2.12) it is easy to show that the solution for ω is approximately

$$\omega^2 \simeq \omega_p{}^2\left(1 + \frac{3k^2\langle v^2\rangle}{\omega_p{}^2} + \cdots\right) \tag{2.14}$$

where $\langle v^2\rangle$ is the mean square velocity of the distribution $f_0(v)$. This is the well-known dispersion equation for plasma oscillations (VLASOV, 1945; LANDAU, 1946; BOHM and GROSS, 1949). For small wave numbers, the frequency is essentially given by (1.1). For wave numbers approaching the Debye wave number $k_D = \omega_p/\langle v^2\rangle^{1/2}$, the frequency rises to $2\omega_p$, but the expansion begins to break down.

The derivation of the dispersion equation (2.14) is hardly rigorous since we arbitrarily specified that a principal value should be taken in (2.11) for the otherwise undefined integral.* To obtain (2.14) in a more rigorous manner, and to see the possibility of damping as expressed in (1.2), a more detailed specification of the problem must be made. We chose to make this specification in terms of an initial-value problem, as was done by LANDAU (1946). The formalism will be outlined in the next section.

3. INITIAL-VALUE PROBLEM

In order to gain a greater understanding of the plasma oscillation phenomena we will consider a more specific problem. We will imagine a plasma with certain equilibrium properties specified by the distribution function $f_0(v)$ subjected at $t = 0$ to an initial small disturbance specified by $f_1(v, x, t = 0)$. The linearized plasma equations (2.8) and (2.9) will then be solved to find the behaviour of the plasma as a function of later times. In this way we will be able to separate clearly effects which are general properties of the plasma itself from effects which are specific to the form of initial disturbance. Again, for simplicity we will treat in detail the one-dimensional motion of a simple one-component plasma in the complete absence of collisions. The appropriate generalizations for two-component plasma (electrons and ions) will be indicated along the way.

The small fluctuating part of the distribution function $f_1(v, x, t)$ and the electric field $E(x, t)$ will be Fourier analysed in space and time in a manner appropriate for an initial-value problem. Their Fourier amplitudes are defined by:

$$\begin{Bmatrix} f_1(v, \omega, k) \\ E(\omega, k) \end{Bmatrix} = \int_0^\infty dt \int_{-\infty}^\infty dx\, e^{-i(kx - \omega t)} \begin{Bmatrix} f_1(v, x, t) \\ E(x, t) \end{Bmatrix} \tag{3.1}$$

and the reciprocal relations:

$$\begin{Bmatrix} f_1(v, x, t) \\ E(x, t) \end{Bmatrix} = \frac{1}{(2\pi)^2}\int_W d\omega \int_{-\infty}^\infty dk \times e^{i(kx - \omega t)} \begin{Bmatrix} f_1(v, \omega, k) \\ (E\omega, k) \end{Bmatrix} \tag{3.2}$$

* The specification of a principal value amounts to ignoring the effects of trapped particles. See BOHM and GROSS (1949) and Sections 7 and 8 below.

where the contour W is chosen in the upper half-plane parallel to the real ω-axis, above any poles in $f_1(v, \omega, k)$ or $E(\omega, k)$.

If the linearized Boltzmann equation (2.8) is multiplied by $\exp(-ikx + i\omega t)$ and integrated over space and time it can be cast in the form:

$$\int_{-\infty}^{\infty} dx \left[f_1(v, x, t)\, e^{-ikx+i\omega t} \right]_{t=0}^{t=\infty} + \int_{-\infty}^{\infty} dx \int_0^{\infty} dt\, e^{-ikx+i\omega t} \times \left[(-i\omega + ikv) f_1 - \frac{e}{m} \frac{\partial f_0}{\partial v} E \right] = 0. \quad (3.3)$$

With the assumption that ω is in the upper half-plane the first term becomes the negative of the spatial transform of the initial disturbance $f_1(v, x, t = 0)$. We define this initial transform as $\Phi(v, k)$:

$$\Phi(v, k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f_1(v, x, t = 0). \quad (3.4)$$

Then with the definitions (3.1) for the transforms of $f_1(v, x, t)$ and $E(x, t)$, equation (3.3) can be solved for $f_1(v, \omega, k)$:

$$f_1(v, \omega, k) = \frac{1}{i(kv - \omega)} \left(\Phi(v, k) + \frac{e}{m} \frac{\partial f_0}{\partial v} E(\omega, k) \right). \quad (3.5)$$

We note that (3.5) differs from our previous expression (2.10) by the presence of the initial value term $\Phi(v, k)$ and the interpretation of ω as a complex variable.

An equation for $E(\omega, k)$ can be obtained by substituting the result for $f_1(v, \omega, k)$ into Poisson's equation (2.9):

$$\left[k^2 - \frac{4\pi e^2}{m} \int \frac{1}{v - \frac{\omega}{k}} \frac{\partial f_0}{\partial v} dv \right] E(\omega, k) = 4\pi e \int \frac{\Phi(v, k)}{v - \frac{\omega}{k}} dv. \quad (3.6)$$

We now define a very important function $H\left(k, \frac{\omega}{k}\right)$:

$$H\left(k, \frac{\omega}{k}\right) = \frac{k^2}{\omega_p^2} - \int \frac{G(v)}{v - \frac{\omega}{k}} dv \quad (3.7)$$

where $G(v)$ is given by (2.12) and ω_p^2 is defined in (1.1). Then equation (3.6) can be written:

$$E(\omega, k) = \frac{4\pi e}{\omega_p^2 H\left(k, \frac{\omega}{k}\right)} \int \frac{\Phi(v, k)}{v - \frac{\omega}{k}} dv. \quad (3.8)$$

The solution for the electric field $E(x, t)$ is then the transform of (3.8):

$$E(x, t) = \frac{e}{\pi \omega_p^2} \int_{-\infty}^{\infty} dk \int_W d\omega\, e^{ikx - i\omega t} \frac{\int \frac{\Phi(v, k)\, dv}{v - (\omega/k)}}{H(k, (\omega/k))}. \quad (3.9)$$

This is the basic result for the electric field, at times $t > 0$, due to an initial disturbance $f_1(v, x, t = 0)$ whose spatial Fourier transform is $\Phi(v, k)$.

If the distribution function $f_1(v, x, t)$ is desired, it is a simple matter to substitute (3.8) for $E(\omega, k)$ into (3.5) and perform the necessary integrations over dk and $d\omega$. The electric field has more direct meaning and we will concentrate on it.

For future reference we record the generalization of these results to a two-component plasma (electrons of charge $-e$, mass m, average density n_0 and ions of charge $+e$, mass M and equal density). If $f_e(v)$ and $f_i(v)$ are the electron and ion equilibrium distributions respectively; $G_e(v)$ and $G_i(v)$, their corresponding normalized derivatives (2.12); and $\Phi_e(v, k)$ and $\Phi_i(v, k)$, the appropriate Fourier transforms of the initial disturbances (3.4), the equations (3.7), (3.8), (3.9) are valid provided one makes the following replacements:

$$\omega_p^2\, G(v) \to \omega_{pe}^2\, G_e(v) + \omega_{pi}^2\, G_i(v) \quad (3.10)$$

$$\Phi(v, k) \to \Phi_e(v, k) - \Phi_i(v, k) \quad (3.11)$$

where ω_{pe}^2 is given by equation (1.1) and $\omega_{pi}^2 = (m/M)\omega_{pe}^2$. For an electrically neutral plasma made up of electrons and several different types of ions, equations (3.10) and (3.11) have obvious generalizations.

4. STRUCTURE OF THE EXPRESSION FOR $E(x, t)$ AND THE DISPERSION EQUATION

The initial-value problem of the preceding section has the solution for the electric field $E(x, t)$ given by (3.9). Since the time dependence is of primary importance, we rewrite (3.9) to exhibit the time dependence of one spatial Fourier component at a time:

$$E(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} E(k, t) \quad (4.1)$$

with

$$E(k, t) = \frac{2e}{\omega_p^2} \int_W d\omega\, e^{-i\omega t} \frac{\int \frac{\Phi(v, k)}{v - (\omega/k)} dv}{H(k, (\omega/k))} \quad (4.2)$$

The temporal behaviour of the plasma disturbance is determined by the structure of integrand in (4.2). We note that it is made up of two parts—one part depending on the equilibrium properties of the plasma $\left[H\left(k, \frac{\omega}{k}\right) \text{ defined by (3.7)} \right]$ and one part depending on the initial conditions [the integral over $\Phi(v, k)$]. To

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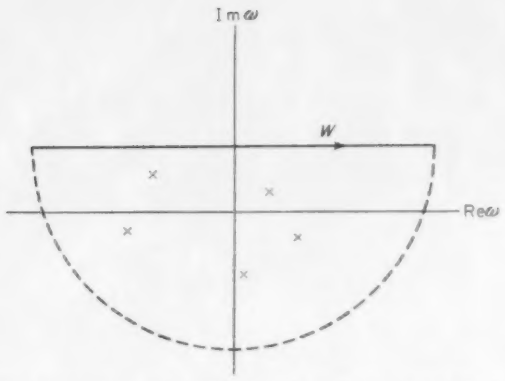

FIG. 1(a).—Complex frequency plane with Laplace transform contour W and poles of integrand of equation (4.2).

see clearly how characteristic frequencies can emerge from (4.2), let us consider how we might go about evaluating the integral. The path of integration W is shown in Fig. 1(a). It is chosen to lie above any poles in the integrand. Because of the exponential time dependence ($t > 0$), the natural thing is to close the contour, as shown by the dotted semicircle, and express the integral in terms of a sum of the residues of the poles of the integrand.

To do this we must have the integrand defined over the whole complex plane. The definition of $H\left(k, \frac{\omega}{k}\right)$ given by (3.7) is valid only in the upper half-plane. Consequently, we must define it below the real axis by analytic continuation. Assuming that this has been done, the expression for $E(k, t)$ can be written:

$$E(k, t) = \frac{4\pi e}{i\omega_p{}^2} \sum_m e^{-i\omega_m t} \text{ Residue} \left[\frac{\int \frac{\Phi(v, k)}{v - \frac{\omega}{k}}\, dv}{H\left(k, \frac{\omega}{k}\right)} \right]_{\omega = \omega_m} \tag{4.3}$$

where the sum over m symbolizes the contributions from the various poles of the integrand shown in Fig. 1(a). If the integrand has a pole in the upper half-plane, there will be in the sum an exponentially increasing term which will dominate the solution after sufficiently long times. If there is a pole on the real axis, there will be a steady-state solution of that frequency; if all the poles are in the lower half-plane, there will be only exponentially decreasing solutions.

We see that the possibility of damped or growing or steady-state oscillations of the plasma is governed by the poles of the integrand of (4.2). For a 'reasonable' initial disturbance ($\Phi v, k$) the integral $\int \frac{\Phi(v, k)}{v - (\omega/k)}\, dv$ will not have poles in the complex plane and the time dependences of $E(k, t)$ will be determined completely by the zeros of $H\left(k, \frac{\omega}{k}\right)$. The equation

$$H\left(k, \frac{\omega}{k}\right) = 0 \tag{4.4}$$

is often called the *dispersion equation* since it yields in principle a solution for the frequency $\omega(k)$ as a function of wave number k. We note that, with the definition (3.7) of H, equation (4.4) is formally identical with equation (2.11) of our elementary discussion. The difference lies in the fact that (3.7) serves to define $H\left(k, \frac{\omega}{k}\right)$ only in the upper half of the complex ω-plane. In order to find the zeros of H in the whole complex plane, we must determine its analytic continuation into the lower half-plane. The function defined by (3.7) is certainly not analytic across the real axis, as can be seen most easily by recalling that*

$$\lim_{\frac{\omega}{k} \to u \pm i\epsilon} \frac{1}{v - \frac{\omega}{k}} = P\left(\frac{1}{v - u}\right) \pm \pi i \delta(v - u) \tag{4.5}$$

where P means Cauchy principal value. This shows that H, defined by equation (3.7), is discontinuous across the real axis, with a jump equal to

$$\Delta H \equiv H\left(\frac{\omega}{k} = u + i\epsilon\right) - H\left(\frac{\omega}{k} = u - i\epsilon\right) = -2\pi i G(u) \tag{4.6}$$

With (3.7) as the definition of $H(\omega)$ for $\text{Im}\,(\omega) > 0$, it is clear that the analytic continuation into the lower half-plane can be accomplished by the following definition:

$$H(\omega) = \begin{cases} \dfrac{k^2}{\omega_p{}^2} - \displaystyle\int \frac{G(v)}{v - \frac{\omega}{k}}\, dv & \text{for Im}\,(\omega) > 0 \\ \dfrac{k^2}{\omega_p{}^2} - \displaystyle\int \frac{G(v)}{v - \frac{\omega}{k}}\, dv - 2\pi i G\left(\frac{\omega}{k}\right) & \text{for Im}\,(\omega) < 0 \end{cases} \tag{4.7}$$

In both integrals the dv integration is along the real v-axis from $-\infty$ to $+\infty$. Equation (4.7) defines $H(\omega)$ over the whole complex plane for the purposes of the

* We assume for definiteness and convenience that k is positive. For k negative the argument can be suitably revised.

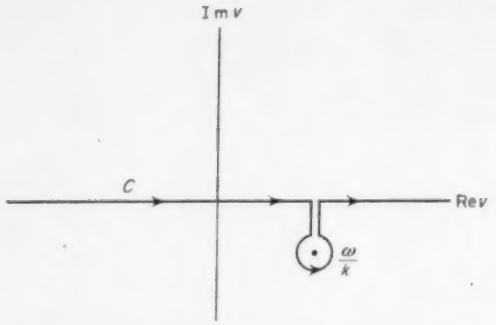

FIG. 1(b).—Complex velocity plane with contour C for the definition of $H(\omega)$ as an analytic function of ω.

dispersion equation (4.4). Following LANDAU (4.7) can be compactly written in terms of a contour integral in the complex v-plane:

$$H(\omega) = \frac{k^2}{\omega_p^2} - \int_C \frac{G(v)\,dv}{v - \frac{\omega}{k}} \tag{4.8}$$

where the contour C is shown in Fig. 1(b). For ω in the upper half-plane the contour is along the real v-axis.

In the next section we will examine some general characteristics of the zeros of $H(\omega)$ and their dependence on the equilibrium properties of the plasma through $G(v)$. The existence of poles in the integrand of (4.2) which depend on the initial conditions will be discussed in Section 8 where the possibility of plasma oscillations of *arbitrary* frequency and wave number is explored.

5. GENERAL RESULTS

The roots of the dispersion equation (4.4) are the characteristic frequencies of oscillation of a plasma whose properties are described by $G(v)$. The disposition of these roots in the complex ω-plane determines whether these are damped, steady-state or growing oscillations. Certain general conclusions can be obtained about the location of these roots without specifying $G(v)$ in complete detail. In this section we will consider some of these general properties.

A. *A single-humped velocity distribution cannot support growing oscillations*

First of all we will show that if the equilibrium velocity distribution $f_0(v)$ (or its generalization for a multi-component plasma implied by (3.10)) has a 'single hump' there can be only damped oscillations in the plasma. More specifically we imagine $f_0(v)$ to vanish at $v = \pm\infty$ and to have a peak at some finite velocity V such that $G(v)$ is positive for all v less than V and negative for all v greater than V, with a zero at

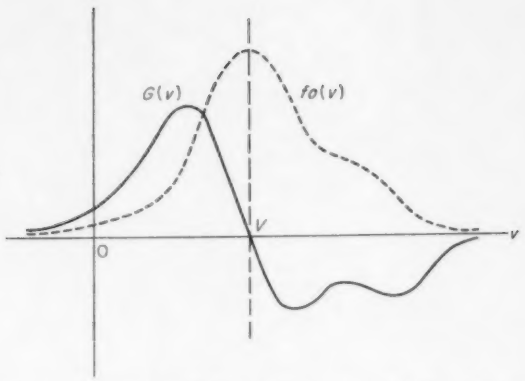

FIG. 2.—Single-humped distribution of velocity and its normalized derivative $G(v)$.

$v = V$. Then we will show that $H(\omega) \neq 0$ for $\mathrm{Im}\,(\omega) > 0$. The general behaviour of $f_0(v)$ and $G(v)$ is shown in Fig. 2.

To prove the assertion made above we consider the real and imaginary parts of $H(\omega)$ for $\omega/k = x + iy$ with $y > 0$. Then the definition (4.7) implies

$$\mathrm{Re}\,(\mathrm{H}) = \frac{k^2}{\omega_p^2} - \int \frac{(v - x)G(v)\,dv}{(v - x)^2 + y^2} \tag{5.1}$$

$$\mathrm{Im}\,(\mathrm{H}) = -y \int \frac{G(v)\,dv}{(v - x)^2 + y^2}. \tag{5.2}$$

In order to have a root in the upper half-plane both real and imaginary part must vanish simultaneously. Suppose that for fixed positive y a value of x can be found so that $\mathrm{Im}\,(H) = 0$. From the form of (5.2) it is apparent that this can always be done. We can now ask whether $\mathrm{Re}\,(H) = 0$ for the same choice (x, y). To show that this is impossible we note that if $\mathrm{Im}\,(H) = 0$ we can add any multiple of it to the right-hand side of (5.1) without changing the value of $\mathrm{Re}\,(H)$. In particular we choose to add $(V - x)/y$ times $\mathrm{Im}\,(H)$ to (5.1). Then by a change of variables from v to $v' = v - V$ in the integral, we obtain

$$\mathrm{Re}\,(H) = \frac{k^2}{\omega_p^2} - \int \frac{v'G(v')\,dv'}{[v' - (x - V)]^2 + y^2} \tag{5.3}$$

By breaking up the range of integration into $v' < 0$ and $v' > 0$, and using the property that $G(v') > 0$ for $v' < 0$ and $G(v') < 0$ for $v' > 0$, equation (5.3) can be written:

$$\mathrm{Re}\,(H) = \frac{k^2}{\omega_p^2} + \int_{-\infty}^{0} \frac{|v'G(v')|}{[v' - (x - V)]^2 + y^2}\,dv' + \int_{0}^{\infty} \frac{|v'G(v')|}{[v' - (x - V)]^2 + y^2}\,dv' \tag{5.4}$$

Since the integrands of both integrals are non-negative everywhere the integrals themselves are necessarily non-negative and $\mathrm{Re}\,(H)$ cannot vanish.

This completes the proof that if $f_0(v)$ is a single humped curve there can be no roots of the dispersion equation in the upper half-plane. Fortunately the above argument cannot be used to exclude roots in the lower

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half-plane as well because the real and imaginary parts of H have added contributions from $\left[-2\pi i G\left(\frac{\omega}{k}\right)\right]$, as shown in (4.7). As a final remark, we note that the converse theorem (a plasma with an equilibrium distribution function having more than one hump *will* support exponentially growing oscillations) is not true in general. In Section 9 we will discuss examples to illustrate this point.

B. *General result for* Im (ω) *provided* Im (ω/k) *is small*

We now consider a general problem in which the velocity distribution $f_0(v)$ is such that the dispersion equation has roots near the real ω-axis. The criterion of nearness to the axis is that Im (ω/k) is small compared to the characteristic spread in velocities involved in $G(v)$. Within this limitation a general result will be obtained for the imaginary part of the frequency in terms of $G(v)$ and the real part of the frequency.

For definiteness we first consider $\mathrm{Im}\left(\frac{\omega}{k}\right) > 0$. Then with $\frac{\omega}{k} = x + iy$, the dispersion equation (4.4) is

$$H = \frac{k^2}{\omega_p^2} - \int \frac{G(v)}{v - x - iy}\, dv. \tag{5.5}$$

We are interested in small (but not vanishing) y. In Appendix 1 it is shown that the Dirac expression (4.5) can be generalized to non-vanishing ε and, in particular, that the integral in (5.5) can be expanded as a power series in y:

$$\int \frac{G(v)\, dv}{v - x - iy} = P\int \frac{G(v)\, dv}{v - x} + \pi i\, G(x) + y\left[iP\int \frac{G'(v)\, dv}{v - x} - \pi G'(x)\right] + \ldots \tag{5.6}$$

where only the first term beyond the $y \to 0^+$ limit is exhibited. With this expansion the real and imaginary parts of H can be written:

$$\mathrm{Re}\,(H) = \frac{k^2}{\omega_p^2} - P\int \frac{G(v)\, dv}{v - x} + \pi y G'(x) + 0(y^2) \tag{5.7}$$

$$\mathrm{Im}\,(H) = -\pi G(x) - yP\int \frac{G'(v)\, dv}{v - x} + 0(y^2). \tag{5.8}$$

It is easily verified that with the definition of H given in (4.7) for Im $(\omega) < 0$, these results hold equally well with y now a negative number.

To lowest order in Im (ω), the zero of $H(\omega)$ occurs when

$$\frac{k^2}{\omega_p^2} = P\int \frac{G(v)}{v - x}\, dv \tag{5.9}$$

and

$$\mathrm{Im}\,(\omega) = \frac{\pi k G(x)}{-P\int \frac{G'(v)\, dv}{v - x}}. \tag{5.10}$$

Equation (5.4) is a dispersion equation for the almost real frequency (actually for Re $(\omega) = kx$) of the sort considered in Section 2, while (5.10) is a result for the small imaginary part of ω. To cast this in a more compact form we will express the integral in the denominator in other terms. Consider the dispersion equation for the complex root $\omega = k(x + iy)$:

$$\frac{k^2}{\omega_p^2} = \int \frac{G(v)}{v - \frac{\omega}{k}}\, dv. \tag{5.11}$$

This equation determines $\omega = \omega(k)$. Differentiation of both sides with respect to k yields:

$$\frac{2k}{\omega_p^2} = \frac{d}{dk}\left[\frac{\omega(k)}{k}\right]\int \frac{G(v)}{\left(v - \frac{\omega}{k}\right)^2}\, dv, \tag{5.12}$$

and an integration by parts leads to the result:

$$\frac{2k}{\omega_p^2} = \frac{d}{dk}\left[\frac{\omega(k)}{k}\right]\int \frac{G'(v)}{v - \frac{\omega}{k}}\, dv \tag{5.13}$$

In the limit of (ω/k) becoming real, the integral becomes a principal-value integral plus a delta function contribution according to (4.5). The principal-value integral is just the integral in the denominator of (5.10). To lowest order in Im (ω), it is sufficient to neglect the delta function term and write

$$-P\int \frac{G'(v)\, dv}{v - x} \simeq -\frac{2k}{\omega_p^2 \frac{d}{dk}\left(\frac{\omega(k)}{k}\right)} \tag{5.14}$$

where $\omega(k)$ is assumed real.

Substituting (5.14) into (5.10) and carrying out the differentiation we obtain a general expression for Im (ω):

$$\mathrm{Im}\,(\omega) \simeq \frac{\pi}{2}\,\omega\left(\frac{\omega_p}{k}\right)^2 G\left(\frac{\omega}{k}\right)\left(1 - \frac{k}{\omega}\frac{d\omega}{dk}\right). \tag{5.15}$$

On the right-hand side, the symbol ω stands for the real part of ω found from the dispersion equation (5.9). Equation (5.15) is, of course, only valid if the right-hand side is small compared to ω.* BOHM and GROSS (1949) obtained an approximate damping constant equivalent to (5.15) with $\omega = \omega_p$.

6. LANDAU DAMPING

As an illustration of our general results of the last section we consider the example treated by LANDAU, the behaviour of stationary electronic plasma in the thermal equilibrium subjected to an initial disturbance. For such a plasma the equilibrium distribution is Maxwellian, and the effective velocity derivative is:

$$G(v) = -\frac{1}{\sqrt{2\pi c^2}}\left(\frac{v}{c}\right)\exp\left(\frac{-v^2}{2c^2}\right) \tag{6.1}$$

where c is the r.m.s. thermal velocity in one dimension ($c^2 = kT/m$).

In Section 2 we have already obtained an approximate solution to (5.9) for the real part of the frequency as a function of wave number, namely (2.14):

$$\omega^2 \simeq \omega_p^2 + 3k^2c^2. \tag{6.2}$$

* It should perhaps be emphasized that there are in general several roots of the dispersion equation. Equation (5.15) gives the imaginary part of the frequency only for roots which happen to lie close to the real axis.

For the imaginary part of the frequency we first note that it is negative according to the theorem about single-humped distributions. Provided the damping is small, we can apply (5.15). With (6.2) for ω it is easy to show that (5.15) becomes:

$$\mathrm{Im}\,(\omega) \simeq -\sqrt{\frac{\pi}{8}}\,\omega_p\left(\frac{\omega_p}{kc}\right)^3 \exp\left(\frac{-\omega^2}{2k^2c^2}\right). \qquad (6.3)$$

This is essentially the LANDAU damping constant (1.2). The only modification is that ω^2 appears in the exponent instead of ${\omega_p}^2$. This means that the somewhat more accurate result (6.3) is a factor $\exp(+3/2) = 4{\cdot}48$ smaller than the LANDAU result (1.2).

The behaviour of the real and imaginary parts of the frequency is shown in Fig. 3. We note that the damping constant vanishes exponentially for long wavelengths so that such plasma oscillations are essentially steady-state. However, if the wave number is as large as one-half of the Debye wave number ($k_D = \omega_p/c$), the damping constant is of the order of $0{\cdot}15\omega_p$ and the oscillations are effectively damped out after six or seven periods.

For larger wave numbers ($k \sim k_D$) the approximate results obtained so far fail to hold. Explicit numerical solution of the dispersion equation is then necessary.

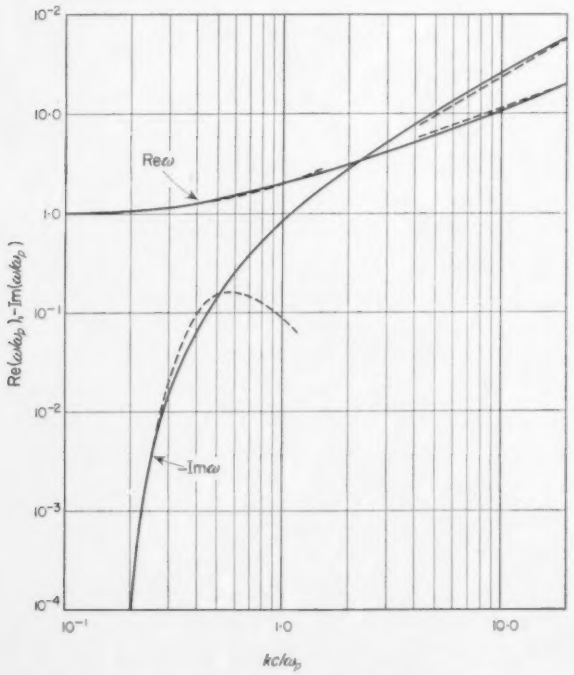

FIG. 3.—Real and imaginary parts of the frequency as a function of wave number for a stationary one-component plasma in thermal equilibrium. The frequency is given in units of ω_p (1.1), while the wave number is expressed in units of the Debye wave number (ω_p/c). The dotted curves represent approximate formulas derived in the text.

With a Gaussian or sum of Gaussians for $f_0(v)$ the tables of FADDEEVA and TERENT'EV (1954) can be employed (the details are presented in Appendix 2). The results of the calculation are shown in Fig. 3 as the solid curves. The real and imaginary parts of the frequency become equal at $k = 2{\cdot}32k_D$. For wave numbers larger than this critical value it becomes meaningless to speak of plasma oscillations in any ordinary sense.

The asymptotic behaviour of the dispersion relation in the region of large wave numbers can be found by utilizing the fact that the imaginary part of the frequency is much larger than the real part. Then the dispersion relation can be expanded in inverse powers of $\mathrm{Im}\,(\omega/k)$. Its real and imaginary parts become:

$$\begin{aligned}\frac{k^2}{{\omega_p}^2} &= -2\pi\,\mathrm{Im}\,[G(x-\mathrm{i}y)] \\ &\quad -\frac{1}{y^2}\left[1-\frac{1}{y^2}(3\langle v^2\rangle - 6x\langle v\rangle + 3x^2)+\ldots\right] \\ 0 &= 2\pi\,\mathrm{Re}\,[G(x-\mathrm{i}y)] - \frac{1}{y^3}\left[2\langle v\rangle - 2x + 0\left(\frac{1}{y^2}\right)\right] \end{aligned} \qquad (6.4)$$

where $\omega = (x - \mathrm{i}y)k$ and $\langle v^n\rangle$ is the average value of v^n for the equilibrium distribution $f_0(v)$. With the assumption that y is large while $\mathrm{Im}\,(G)$ is not, (6.4) is approximately:

$$\begin{aligned}\frac{k^2}{{\omega_p}^2} + 2\pi\,\mathrm{Im}\,[G(x-\mathrm{i}y)] &\simeq 0 \\ \mathrm{Re}\,[G(x-\mathrm{i}y)] &\simeq 0\end{aligned} \qquad (6.5)$$

Using the specific Maxwellian form (6.1), the second equation in (6.5) reduces to $(c/y)^2 = \left(\frac{c^2}{xy}\right)\tan\left(\frac{xy}{c^2}\right)$. For small (c/y), this equation is satisfied by $xy \simeq n\pi c^2$ ($n = 1, 2, \ldots$). Restricting our attention to the smallest root (see Appendix 2), the first equation in (6.5) yields y implicitly:

$$\left.\begin{aligned}\frac{y}{c}\exp\left(\frac{y^2}{2c^2}\right) &\simeq \frac{1}{\sqrt{2\pi}}\left(\frac{kc}{\omega_p}\right)^2 \\ \text{and}\qquad \frac{x}{c} &\simeq \frac{\pi c}{y}\end{aligned}\right\} \qquad (6.6)$$

Equations (6.6) are the results for the real and imaginary parts of ω for large wave number first obtained by LANDAU (1946). An approximate solution of (6.6) is:

$$\begin{aligned}\mathrm{Re}\,(\omega) &\simeq \pi\frac{kc}{\sqrt{\ln\left(\frac{k^4c^4}{2\pi{\omega_p}^4}\right)}} \\ \mathrm{Im}\,(\omega) &\simeq -kc\left\{\sqrt{\ln\left(\frac{k^4c^4}{2\pi{\omega_p}^4}\right)} - \frac{1}{2}\frac{\ln\left[\ln\left(\frac{k^4c^4}{2\pi{\omega_p}^4}\right)\right]}{\sqrt{\ln\left(\frac{k^4c^4}{2\pi{\omega_p}^4}\right)}}\right\}\end{aligned} \qquad (6.7)$$

The limiting expressions (6.7) show that both the real and imaginary parts of ω increase with increasing k but that the ratio of the imaginary to real parts increases logarithmically with k. These asymptotic forms are also plotted in Fig. 3.

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As mentioned earlier, in this region of large wave number the damping is so great that the concept of oscillations in a plasma loses its validity. Organized collective motion cannot be maintained for wavelengths small compared to the Debye length. It is more appropriate instead to describe the behaviour of the plasma in terms of the individual particles, taking full account of the collisions. It is perhaps fortunate that the simple theory of plasma oscillations without collisions itself shows that oscillations can exist only for long wavelengths. It is clear that if it had predicted real oscillations at short wavelengths the result would be suspect because of the neglect of collisions.

7. PHYSICAL MECHANISMS FOR THE DAMPING OR GROWING OF OSCILLATIONS

In Section 5 we derived a general result (5.15) for the imaginary part of the frequency in terms of the properties of the equilibrium distribution through $G(v)$. In the present section we will discuss the physical mechanism ('trapping') which causes the damping or growing of plasma oscillations implied by (5.15) and give a semi-quantitative derivation of (5.15) based on physical arguments. In addition to the 'trapping' mechanism, there exists another physically different process ('bunching') by which plasma oscillations can grow or decay. In the hydrodynamic approximation only the bunching mechanism occurs. Bunching will be discussed qualitatively at the end of this section.

In a plasma with a certain low level of plasma oscillations present there is a definite (small) amount of energy in the organized motion of oscillation and a much larger amount of energy in disorganized thermal motion. For a steady-state oscillation the division of energy between organized motion (plasma energy) and disorganized motion (particle energy) stays constant. If the oscillation is damped or growing there must be an increase of the particle or plasma energy at the expense of the other. As pointed out by Bohm and Gross, one mechanism of this energy transfer is the trapping of particles by the electric field of the oscillation into the moving potential troughs.

If we think of a small amplitude plasma wave of a certain wavelength and phase velocity it is evident that only particles moving with velocities approximately equal to the phase velocity can be caught in a stable position by the wave and be carried along by it. If the velocity difference between wave and particle is too great the particles will slide over the crests and troughs without any change in energy on the average. To be more specific, consider a simple electronic plasma with an equilibrium velocity distribution $f_0(v)$. At $t = 0$, a plasma wave is established with electric field $E(x, t) = E_0 \sin(kx - \omega t)$. The wave travels with a phase velocity $u = \omega/k$.

In the co-ordinate system moving along with the wave, particles with a laboratory velocity approximately equal to u will be slowly moving and will experience a time independent force $F(x) = -eE_0 \sin kx$. For small displacements this approximates a linear restoring force. The particles will tend to begin oscillating with a frequency:

$$\omega_{\text{osc}} \simeq \sqrt{\frac{eE_0 k}{m}} \tag{7.1}$$

provided their relative velocity is not too great. Since the change in potential energy from peak to valley in the wave is $\Delta V = 2eE_0/k$, the maximum relative velocity Δv is roughly

$$\tfrac{1}{2} m(\Delta v)^2 \sim \frac{eE_0}{k}. \tag{7.2}$$

Particles with velocities differing from the phase velocity by more than Δv will escape from their initial cycle of the wave and run over the wave crests. Equation (7.2) represents some sort of average over the various initial positions in x at which a particle may be found.

We see that of all the particles in the distribution $f_0(v)$ those with velocities in the range $(u - \Delta v) < v < (u + \Delta v)$ will be caught by the wave and carried along with it. The trapping time is of the order ω_{osc}^{-1} since the particles presumably need to undergo roughly one cycle of oscillation before they know that they have been caught. In this process of trapping the particles undergo a change in energy. Before trapping, each particle of velocity v had an energy $\frac{1}{2}mv^2$. After trapping its energy is $\frac{1}{2}m[u^2 + (v - u)^2]$, where the first part is the energy in the absence of oscillation about equilibrium and the second term is the energy of oscillation. The energy change of the particles per unit volume due to trapping can be calculated:

$$\Delta E = \frac{m}{2} \int_{u - \Delta v}^{u + \Delta v} f_0(v) \left[u^2 + (v - u)^2 - v^2\right] dv. \tag{7.3}$$

If we expand $f_0(v)$ around $v = u$ it is easy to show that ΔE is approximately:

$$\Delta E \simeq -\tfrac{2}{3} mu(\Delta v)^3 n_0 G(u) \tag{7.4}$$

where n_0 is the number of particles per unit volume and $G(u)$ is defined by (2.7). The result (7.4) is qualitatively reasonable; for example, if there are

more particles moving slightly faster than the phase velocity than are moving slower [$G(u) > 0$], the trapping will tend to slow down more particles than it speeds up and there will be a net decrease in total energy.

The energy change (7.4) is a change in energy of the particles. This must be equal and opposite to the change in energy of the plasma oscillations. Since the change occurs in a time interval of order ω_{osc}^{-1} we can write the order of magnitude equation for the rate of change of plasma energy:

$$\frac{dU}{dt} \simeq -\omega_{\text{osc}}\,\Delta E \tag{7.5}$$

where

$$U = E_0^2/8\pi \tag{7.6}$$

is the energy per unit volume in plasma oscillations. With the definitions (7.1) and (7.2) the expression for ΔE can be written in terms of U so that (7.5) becomes:

$$\frac{dU}{dt} \simeq \left[\frac{8\sqrt{2}}{3}\,\omega\left(\frac{\omega_p}{k}\right)^2 G\left(\frac{\omega}{k}\right)\right] U. \tag{7.7}$$

For fields varying as $\exp(-i\omega t)$ the coefficient of U on the right-hand side of (7.7) is identifiable as $2\,\text{Im}(\omega)$. Consequently we find:

$$\frac{\text{Im}\,(\omega)}{\omega} \simeq \frac{4\sqrt{2}}{3}\left(\frac{\omega_p}{k}\right)^2 G\left(\frac{\omega}{k}\right). \tag{7.8}$$

This is essentially the result (5.15) apart from numerical factor of order unity and the factor $\left(1 - \frac{k}{\omega}\frac{d\omega}{dk}\right)$. This last factor is necessary to make (5.15) invariant under Galilean transformations. Its absence in our present physical derivation can be traced to the fact that in writing down equation (7.5) for the energy change in the plasma we ignored the possibility of energy flow. This is only valid if our 'laboratory' frame of co-ordinates is chosen so that the group velocity of the wave $E_0 \sin(kx - \omega t)$ vanishes. In this frame (7.8) agrees with (5.15). For a multi-component plasma it can be readily seen that the same physical arguments yield (7.8) with the appropriate generalization of $G(v)$ given by (3.10). The derivation of (7.8) shows that the slow damping or growing of oscillations can be attributed to energy exchange between plasma and particles due to trapping of the particles by the wave. Our discussion may be thought somewhat artificial since we postulated that a travelling wave was applied to a plasma so that the particles suddenly had to adapt to a new environment. This is a legitimate initial-value problem, and it presumably has more general application since collisions (although ignored here in describing the plasma oscillations) will tend to feed particles into and out of a velocity band near the phase velocity with random phases relative to the moving wave.

Although the discussions in this report deal with the linearized theory of infinitesimal amplitude plasma oscillations it is worthwhile to remark on the range of validity of our arguments when we imagine the amplitude of oscillations to increase more and more. For the preceding arguments on energy exchange to hold, there must be very little energy in plasma oscillations relative to the particle energy content. Otherwise the plasma will take out or supply large enough amounts of energy so that the 'equilibrium' particle distribution will be distorted significantly. The criterion for validity is that the trapping time ω_{osc}^{-1} shall be long compared to the damping (or growing) time $[\text{Im}\,(\omega)]^{-1}$:

$$\omega_{\text{osc}} \ll |\text{Im}\,(\omega)| \tag{7.9}$$

This sets a limit on the size of the electric field of the plasma oscillations:

$$\left.\begin{aligned} &E_0 \ll E_{\max} \\ \text{where}\quad &E_{\max} = \frac{m\omega^2}{ek}\left(\frac{\omega_p}{k}\right)^4 \left|G\left(\frac{\omega}{k}\right)\right|^2 \end{aligned}\right\} \tag{7.10}$$

The most severe limit comes from the shortest wavelengths. With k equal to the Debye wave number ($k_D = \omega_p/c$), $E_{\max}$ becomes

$$E_{\max} = \frac{mc^2}{e}\,k_D\left(\frac{\omega}{\omega_p}\right)^2 \left|c^2 G\left(\frac{\omega}{\omega_p}\,c\right)\right|^2. \tag{7.11}$$

This sets an upper limit on field strength* such that the potential energy change from peak to valley of the wave is of the order of, or perhaps much less than the thermal energy (depending on the dimensionless factor involving $|G|^2$). For waves of larger amplitude the linearized theory fails and it can be inferred that such waves may involve appreciable numbers of trapped particles and will behave in more complex ways than found here.

The trapping mechanism for growth or decay depends on the presence of particles travelling at or near the phase velocity of the wave, that is, it depends on a distribution of velocities of the plasma particles. There exists another, often dominant, process for instabilities of plasma oscillations. This process can be called *bunching* and does not depend on a finite spread of velocities. It is familiar to electrical engineers in the description of klystrons, travelling wave tubes, etc., and is best discussed within the hydrodynamic approximation.

The bunching process can be understood by considering one or more beams of charged particles moving with certain mean velocities together with a harmonic plasma wave of definite phase velocity. The particles move over the potential wave alternately gaining and losing speed as they go. If a steady-state

* If the wave is damped the limiting value is the initial value of the field, whereas for a growing wave the limiting value is the field strength beyond which the small oscillation approximation fails.

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situation exists, the continuity equation (2.3) shows that the product (nv) is constant. Hence a velocity modulation leads to a density modulation in space (bunching). Provided the resulting charge density is consistent with the potential wave, the steady-state will continue.

One characteristic of bunching is that growing oscillations exist only for wave numbers within some definite range, usually less than some maximum value. To see how such a sharp cut-off can occur, we consider an over-simplified situation with a single beam of electrons (charge $-e$, mass m) of equilibrium density n_0 and mean velocity v_0 in the rest frame of a plasma wave described by a small potential $\phi(x) = \phi_0 \cos(kx)$. From Poisson's equation the charge density necessary to maintain the potential is

$$\rho_P = \frac{k^2}{4\pi}\phi. \tag{7.12}$$

Using conservation of energy to determine the velocity modulation and the continuity equation to convert to a density modulation, we find that the charge density due to bunching is

$$\rho_B = \frac{e^2 n_0}{m v_0^2}\phi = \frac{\omega_p^2}{4\pi v_0^2}\phi. \tag{7.13}$$

We note that ρ_B is independent of wave number while ρ_P is proportional to k^2. It seems clear that bunching can produce more than the necessary charge density to maintain the wave provided k is less than some maximum value. Thus growing will occur only for $k < k_{\max} = \omega_p/v_0$. The electrons are moving too fast to give sufficient bunching for shorter wavelengths.

The above argument is actually fallacious since with only *one* stream of particles we can always find a reference frame where (7.12) equals (7.13) for *any* value of k. This means that there is always a steady-state solution whose frequency in the rest frame of the particles is just ω_p, a result already found in Section 2.

To remedy this flaw in the argument we consider the more realistic two-stream problem by adding a stream of 'ions' (charge $\pm e$, mass M) of the same mean density and having mean velocity $V_0 = v_0 - V$. Then the charge density due to bunching becomes:

$$\rho_B = \frac{\omega_{pe}^2}{4\pi}\left(\frac{1}{v_0^2} + \frac{m}{MV_0^2}\right)\phi. \tag{7.14}$$

For fixed relative velocity V the quantity $4\pi\rho_B/\phi$ (7.14) is sketched in Fig. 4 as a function of v_0 (the negative of the phase velocity of the wave in the rest frame of the electrons). Typical horizontal lines representing $(4\pi\rho_P/\phi)$ (7.12) for two values of wave number are also shown. It will be seen that for k large enough there are four real roots to the equation $\rho_B = \rho_P$, but for small k two of the roots disappear. This means that there are instabilities, as can be inferred by noting that $\rho_B > \rho_P$. The maximum wave number $k_{\max}$ for which growing waves occur is given by the minimum of (7.14) as a function of v_0, i.e. at $mv_0^3 = MV_0^3$. This yields the result:

$$k_{\max} = \left(1 + \left(\frac{m}{M}\right)^{1/3}\right)^{3/2}\frac{\omega_{pe}}{V} \tag{7.15}$$

The two-stream instability will be discussed in detail in Section 9 where the effects of thermal motion will be included. The hydrodynamic limit is illustrated there in Figs. 5(e), (f), and (g).

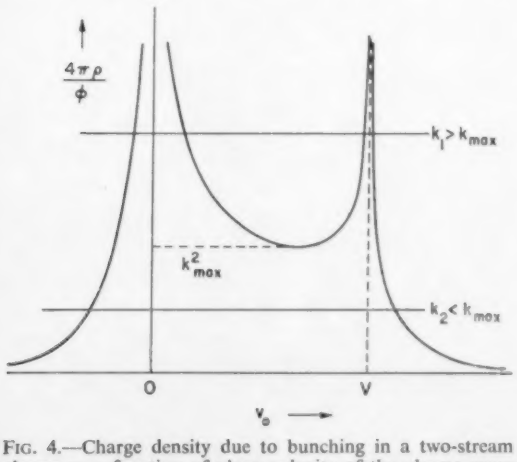

FIG. 4.—Charge density due to bunching in a two-stream plasma as a function of phase velocity of the plasma wave $(-v_0)$ for fixed velocity difference V between the streams. The two horizontal lines represent the charge density necessary to maintain the wave for two different values of wave number.

8. ARBITRARY STEADY-STATE PLASMA OSCILLATIONS

In the preceding sections we have described plasma oscillations in terms of an initial-value problem and have seen that a plasma will support growing or damped waves with a *definite relationship* between wave number and (complex) frequency for a given equilibrium distribution function. For the simple plasma in thermal equilibrium the characteristic plasma frequency ω_p (1.1) and LANDAU damping (1.2) were obtained. On the other hand, VAN KAMPEN (1955) has shown that steady-state oscillatory solutions with *arbitrary* k and ω can be found to satisfy the plasma equations of motion. BERNSTEIN, GREENE and KRUSKAL (1957) show how to construct arbitrary large amplitude wave forms of any given phase velocity by causing just the correct distribution of electrons and ions to be trapped in the finite amplitude potential troughs. In this section we wish to establish

the relationship between our previous results and the apparently contradictory solutions with arbitrary k and ω.* We will see that VAN KAMPEN's waves are contained in our initial-value problem along with the more customary solutions.† The difference, as indicated by the work of BERNSTEIN, GREENE and KRUSKAL, lies in the initial disturbance applied to the plasma. Sufficiently clever initial conditions on the number of trapped and untrapped particles will allow the creation of a plasma wave of arbitrary k and ω. Less ingenious initial conditions will lead to the usual behaviour of the plasma after a short transient period.

First a brief outline of VAN KAMPEN's approach will be given for a simple plasma. He writes down the usual expression (2.10) for the linearized amplitude $f_1(v, \omega, k,)$ for arbitrary k and ω:

$$f_1 = \frac{1}{\mathrm{i}(kv - \omega)} \frac{eE}{m} \frac{\partial f_0}{\partial v} \tag{8.1}$$

Then he observes that for real k and ω the factor $\dfrac{1}{kv - \omega}$ is not defined in the neighbourhood of the phase velocity (ω/k). In the initial-value problem the Dirac replacement (4.5) is appropriate for almost real ω, but VAN KAMPEN chooses to replace (8.1) by the expression:

$$f_1 = \frac{1}{\mathrm{i}\,k} \frac{eE}{m} \frac{\partial f_0}{\partial v} \left[P\left(\frac{1}{v - \dfrac{\omega}{k}} \right) + \lambda(k, \omega)\, \delta\left(v - \frac{\omega}{k} \right) \right] \tag{8.2}$$

where the function $\lambda(k, \omega)$ is to be determined so that Poisson's equation (2.9) is satisfied for the arbitrarily chosen k and ω. A simple calculation shows that $\lambda(k, \omega)$ is given by:

$$\lambda(k, \omega)\, G\left(\frac{\omega}{k}\right) = \frac{k^2}{\omega_p{}^2} - P \int \frac{G(v)}{v - \dfrac{\omega}{k}}\, dv. \tag{8.3}$$

The plasma oscillation represented by (8.2) and (8.3) is a perfectly acceptable solution of the linearized problem. It shows that steady-state waves of arbitrary frequency and wave number can be established in a plasma. That plasmas usually show less arbitrary behaviour can be traced to the fact that the distribution function (8.2) represents initial conditions which are perhaps difficult to establish. The first part of (8.2) shows that the disturbance at velocities away from the phase velocity is made to depend on the properties of the equilibrium distribution. The principal value symbol means that the velocity distribution is not altered at exactly the phase velocity. The delta function term in (8.2) specifies the disturbance to the equilibrium velocity distribution exactly at the phase velocity. The gap implied by the principal value symbol and the delta function are mathematical idealizations in the linearized theory. For a small but finite amplitude wave, particles in a small velocity range Δv (7.2) around the phase velocity are trapped by the wave, while particles outside this interval cannot be trapped. For a wave of negligible amplitude it is clear that the first term in (8.2) represents the initial excess distribution of untrapped particles, while the delta function term specifies the numbers of trapped particles. Consequently we see that VAN KAMPEN's steady-state solution is achieved by carefully choosing the numbers of trapped and untrapped particles according to (8.2) and (8.3).

To obtain the linearized equivalent of the arbitrary wave forms of BERNSTEIN, GREENE and KRUSKAL, it is only necessary to superpose a set of VAN KAMPEN's waves all having the same phase velocity:

$$f_1(v, x, t) = \int dk\, f_1(v, \omega = ku, k)\, e^{\mathrm{i}kx - \mathrm{i}kut} \tag{8.4}$$

where $f_1(v, \omega, k)$ is given by (8.2) and u is the phase velocity. If the arbitrary potential is $\phi(x)$ in the rest frame of the wave form, then using (8.2) and (8.3) equation (8.4) for the required electron distribution can be written:

$$f_1(v, x) = \frac{-en_0}{m} \phi(x) \left[G(v) P\left(\frac{1}{v}\right) - \delta(v) P \int \frac{G(v')}{v'}\, dv' \right] + \frac{1}{4\pi e} \frac{d^2\phi}{dx^2}\, \delta(v) \tag{8.5}$$

There remains the question of obtaining VAN KAMPEN's solutions from the initial-value problem of Sections 3 and 4. The steady-state solution (8.2) can be written, apart from constants, as:

$$f_1 e^{\mathrm{i}k_0 x - \mathrm{i}\omega_0 t} = -\mathrm{i} \left[\frac{G(v)}{v - \dfrac{\omega_0}{k_0} - \mathrm{i}\epsilon} + H\left(\frac{\omega_0}{k_0}\right) \delta\left(v - \frac{\omega_0}{k_0} \right) \right] e^{\mathrm{i}k_0 x - \mathrm{i}\omega_0 t} \tag{8.6}$$

where $H(\omega/k)$ is defined by (3.7) and (4.7). We have labelled the arbitrary parameters as k_0, ω_0. To show that (8.6) is contained in the initial-value formalism it is sufficient to prove that the initial disturbance implied by (8.6) gives rise to steady-state oscillations of the correct frequency. We saw in Section 4 that the time behaviour of the oscillations was determined by the residues in the complex ω-plane of the function:

$$Z(\omega) = \frac{\displaystyle\int \frac{\Phi(v, k)}{v - \dfrac{\omega}{k}}\, dv}{H\left(k, \dfrac{\omega}{k}\right)} \tag{8.7}$$

* We restrict the discussion to small oscillations. VAN KAMPEN's solutions can be shown to be a limiting case of the finite amplitude solutions of BERNSTEIN, GREENE, and KRUSKAL.

† For a detailed mathematical discussion of the equivalence of the initial-value problem and the VAN KAMPEN treatment, see CASE (1959).

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where

$$\Phi(v, k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f_1(v, x, t = 0) \qquad (8.8)$$

is the Fourier transform of the initial disturbance. In the discussion below equation (4.3) the emphasis was on poles of $Z(\omega)$ caused by zeros of the function $H(\omega)$. Now, however, there are singularities in the numerator. With (8.6) evaluated at $t = 0$ in (8.8) we find:

$$\Phi(v, k) = -2\pi i\, \delta(k - k_0) \times \left\{ \frac{G(v)}{v - \frac{\omega_0}{k_0} - i\epsilon} + H\left(\frac{\omega_0}{k_0}\right) \delta\left(v - \frac{\omega_0}{k_0}\right) \right\} \qquad (8.9)$$

and

$$\int \frac{\Phi(v, k)\, dv}{v - \frac{\omega}{k}} = 2\pi i\, \delta(k - k_0) \frac{H\left(\frac{\omega}{k}\right)}{\frac{\omega}{k} - \frac{\omega_0}{k_0} - i\epsilon} \qquad (8.10)$$

This leads to the function $Z(\omega)$:

$$Z(\omega) = 2\pi i\, \delta(k - k_0) \frac{1}{\frac{\omega}{k} - \frac{\omega_0}{k_0} - i\epsilon}. \qquad (8.11)$$

We see that the only pole remaining in $Z(\omega)$ is one at $\omega = \omega_0$ with residue $2\pi i\, \delta(k - k_0)$. Thus the electric field will oscillate with a frequency $\omega = \omega_0$, as required. A simple calculation using equations (3.2), (3.5) and (3.8) gives the distribution function (8.6) with the correct time dependence. Equation (8.10) shows that the initial distribution is so chosen that all the poles apparently present in $Z(\omega)$ from the vanishing of its denominator are eliminated by zeros in the numerator. In addition the initial disturbance provides a pole in $Z(\omega)$ at just the right place to yield a steady-state solution. If the initial disturbance deviates from its carefully chosen form, then the time behaviour of the plasma is again governed by the dispersion equation (4.4) and the character of the oscillations is determined by the equilibrium properties of the plasma, not by the initial perturbation.

9. CRITERION FOR GROWING PLASMA OSCILLATIONS

It has been known for a long time that under certain circumstances plasmas will support growing oscillations. Examples are the well-known double-stream amplifier (BOHM and GROSS, 1949; HAEFF, 1949; NERGAARD, 1948; PIERCE and HEBENSTREIT, 1949), described qualitatively in Section 7, in which two electron streams with very narrow spreads in velocity move through each other with parallel but different velocities, and the generalization of this proposed by BUNEMAN (1958) in which electrons and ions undergo relative motion. Generally these examples are discussed in terms of the hydrodynamic approximation in which the velocity distribution $f_0(v)$ is a sum of delta functions. Then the solution of the dispersion equation reduces to finding the roots of a finite polynomial in ω. For most practical purposes the idealized problem is an adequate representation. But under some conditions it is useful to know something about the actual problem in which realistic velocity distributions occur.

For instance in the double stream amplifer the idealized solution implies that there are complex roots for ω with positive imaginary part (growing waves) as long as there is a velocity difference between the two streams, no matter how small that difference becomes. In an actual system, however, the beams will have a finite velocity spread. For velocity differences smaller than some velocity of order the spread of the beams, the velocity distribution $f_0(v)$ will become a 'single-humped curve' in the sense of Section 5. The theorem proved there shows that then growing waves cannot exist. It may be of practical interest to establish this minimum velocity difference. It is of particular relevance for BUNEMAN's discussion of the conductivity of a plasma since he suggests that the relative motion of electrons and ions accelerated in opposite directions by an applied electric field allows the plasma to support growing plasma oscillations which absorb energy from particles and so provide a mechanism for resistivity. For a hot plasma there is a considerable spread in velocities and consequently an appreciable threshold of relative velocity between electrons and ions below which the plasma cannot support growing waves. This will presumably complicate BUNEMAN's mechanism for plasma resistivity.

In this section we will discuss the criterion for growing waves in a plasma and will illustrate it with simple examples. It will be convenient to think of the equilibrium velocity distribution as initially a 'single-humped' distribution which is then distorted in shape until plasma oscillations with growing exponents are possible. In terms of the roots of the dispersion equation (4.4), all the roots will be initially in the lower half of the complex ω-plane. As the equilibrium velocity distribution is changed into a more complicated double (or multi) humped curve the roots move upward until some root lies on the real axis. Further suitably chosen distortion puts one or more roots in the upper half-plane and the realm of growing oscillations has been reached. The critical case is a root on the real axis for some wave number k. The

conditions for a root on the real axis can be found by specializing equations (5.7) and (5.8) to $y = 0$:

$$\left.\begin{aligned} \mathrm{Re}\,(H) &= \frac{k^2}{\omega_p{}^2} - P\int \frac{G(v)}{v-x}\,dv = 0 \\ \mathrm{Im}\,(H) &= -\pi G(x) = 0 \end{aligned}\right\} \tag{9.1}$$

where $x = \omega/k$ is the real phase velocity. The second equation in (9.1) shows that $G(x) = 0$ i.e. the phase velocity of the wave is at a position of zero slope of the distribution function. This is, of course, consistent with our general result (5.15) for $\mathrm{Im}\,(\omega)$. For a single-humped distribution the integral in the first equation of (9.1) is always negative so that $\mathrm{Re}\,(H) \neq 0$, as was shown explicitly in equation (5.4). If we imagine a suitable distortion of the initial distribution the integral will become less and less negative until it finally vanishes. Then (9.1) will have a solution for $k = 0$. Further modification can give roots on the real axis for $k > 0$. For any chosen k value the criterion for the onset of growing waves is:

$$P\int \frac{G(v)}{v-x}\,dv = \frac{k^2}{\omega_p{}^2}, \quad G(x) = 0. \tag{9.2}$$

With $G(x) = 0$ the integral is not a principal value integral, but is an ordinary well-behaved integral.

It is of some interest to establish the connexion between our criterion (9.2) and the condition for instability of servomechanisms, known in electric circuit theory as NYQUIST's criterion.* In the upper half of the complex ω/k plane, the function $H(\omega/k)$ given by (4.7) is an analytic function with certain zeros, which characterize the plasma oscillations, and with no poles or other singularities. Consider the contour C in the $z = \omega/k$ plane shown in Fig. 5(a).

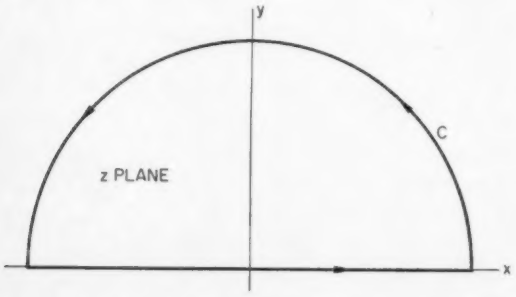

FIG. 5(a).—Complex $z = \omega/k$ plane and the contour C on and inside of which $H(z)$ is analytic and perhaps possesses zeros.

The contour consists of the real z-axis and a circle at infinity in the upper half-plane. Along the real axis, the real and imaginary parts of $H(z)$ are given by the left-hand sides of equation (9.1). On the great circle H is constant and equal to $(k^2/\omega_p{}^2)$. If we now define $w = H(z)$, each point on the contour C corresponds to a point in the w-plane. Consequently the curve C can be mapped into a closed curve D in the complex

* The author is indebted to Dr. B. D. FRIED for pointing out the relevance of NYQUIST's theorem to the present problem.

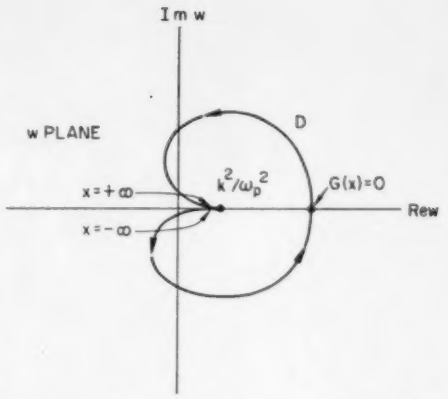

FIG. 5(b).—Map of the contour C on to the complex plane $w = H(z)$. This is called a NYQUIST diagram.

w-plane. Such a mapping is shown schematically in Fig. 5(b). From equation (9.1) it is easy to see qualitatively how the contour D will look. In particular, the one shown in Fig. 5(b) corresponds to a single-humped velocity distribution (Fig. 2).

The behaviour of the contour D in the w-plane is the basis of the NYQUIST criterion for instability. As a simple consequence of Cauchy's theorem the number of zeros of $H(z)$ inside the contour C is given by:

$$N_0 = \frac{1}{2\pi i}\int_C dz\,\frac{H'(z)}{H(z)}. \tag{9.3}$$

In terms of w and the contour D this is transformed into

$$N_0 = \frac{1}{2\pi i}\int_D \frac{dw}{w}. \tag{9.4}$$

By Cauchy's theorem, if the closed contour D encircles the origin one or more times, N_0 will not be zero and $H(z)$ will have roots in the upper half-plane, i.e. growing oscillations. This is NYQUIST's theorem (NYQUIST, 1932) and the plot of the curve D is called a NYQUIST diagram. Fig. 5(c) contains a series of schematic NYQUIST diagrams for a fixed value of wave number, along with corresponding $G(v)$ curves. The sequence of $G(v)$ curves represents a set of velocity distributions progressing from a single-humped distribution through to a distribution with two well-separated peaks. The first three NYQUIST diagrams imply no growing oscillations for that particular wave number, while the last indicates instability.

We see from equation (9.4) or Fig. 5(c) that the critical condition for passing into or out of the region of growing waves is that the curve D pass through the origin of w i.e. the real and imaginary parts of $H(z)$ vanish for some real $z = x$. This is just our criterion given by equation (9.2).

As a first illustration of equation (9.2) we consider a simple electronic plasma of two streams of electrons of the same density with equal and opposite mean velocities u. The velocities in each stream are smeared out by an amount characterized by the parameter Δ. We will assume that the distribution function $f_0(v)$ has the following form:

$$\frac{1}{n_0}f_0(v) = \frac{\Delta}{\pi}\left[\frac{1}{(v-u)^2+\Delta^2} + \frac{1}{(v+u)^2+\Delta^2}\right]. \tag{9.5}$$

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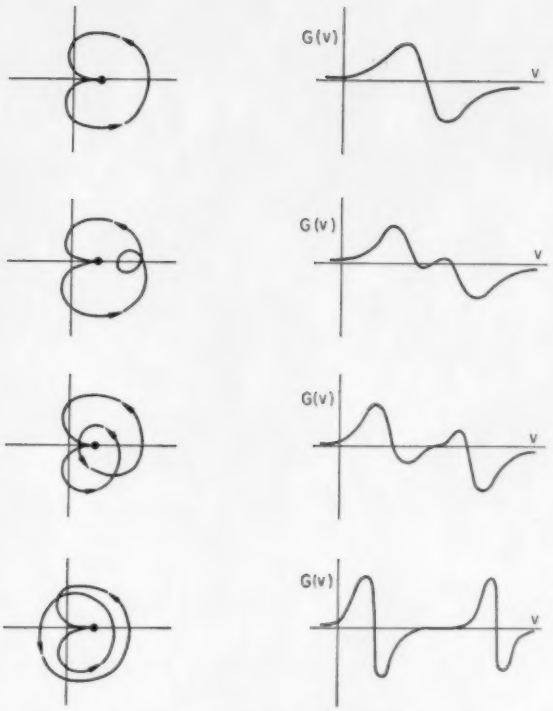

FIG. 5(c).—A series of NYQUIST diagrams and the corresponding derivatives of the velocity distribution function. The first three diagrams indicate no growing oscillations, while the last one shows a growing instability.

Our problem will be to find the minimum velocity difference $[2u(k)]$ such that growing waves can occur for fixed wave number k. For $u < \Delta/\sqrt{3}$, equation (9.5) represents a single-humped curve. Consequently we expect a larger value of u to be necessary for growing oscillations, even for vanishing wave number.

From the symmetrical form of $f_0(v)$ it follows that $G(x) = 0$ only for $x = 0$. Then the other part of equation (9.2) reduces to

$$P\int \frac{G(v)}{v}\,dv = \frac{k^2}{\omega_p{}^2}. \tag{9.6}$$

With equation (9.5) for $f_0(v)$ the integral is an elementary one. The result is:

$$\frac{2(u^2 - \Delta^2)}{(u^2 + \Delta^2)^2} = \frac{k^2}{\omega_p{}^2} \tag{9.7}$$

Clearly, for $u < \Delta$ the left-hand side is negative and no growing waves can exist. $u = \Delta$ is the critical velocity such that waves with zero wave number begin to grow. This is appreciably greater than the velocity for which the distribution becomes a double-humped curve. For $u > \Delta$ there is a range of wave numbers from $k = 0$ up to some $k = k_{\max}$ for which growing oscillations occur. The left-hand side of (9.7) has its maximum value of $0{\cdot}25/\Delta^2$ at $u = \sqrt{3}\Delta$. Consequently the largest wave number to have growing oscillations is $k = 0{\cdot}5\omega_p/\Delta$.* Fig. 5(d) shows how $k_{\max}$ varies as a function of the velocity difference $(2u)$. For very high velocity differences (9.7) yields $k_{\max} = \sqrt{2}\omega_p/u$, a result that is well known for double-stream amplifiers (HAEFF, 1949) and is independent of the detailed shape of the distribution (9.5).

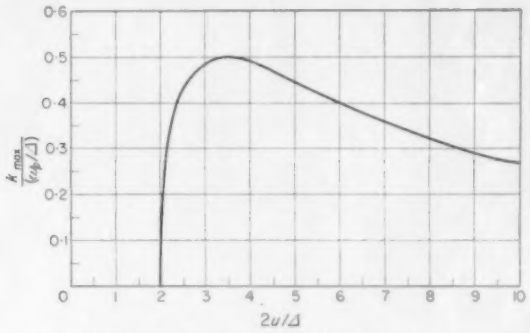

FIG. 5(d).—Range of wave numbers $(0 < k < k_{\max})$ for which growing oscillations can occur as a function of mean velocity difference for two streams of electrons with resonance shaped velocity spreads. The mean velocity difference $(2u)$ is given in units of the resonance parameter Δ and the maximum wave number is expressed in units of the equivalent Debye wave number (ω_p/Δ).

At this point we might mention an observation by Gell-Mann that the distribution (9.5) is so simple that it is just as easy to solve the dispersion equation (4.4) directly for the *complex* roots. The reader may verify that the results summarized in Fig. 5(d) follow from the explicit solution of the dispersion equation.

As a second illustration, typical of thermonuclear problems, we consider a two component plasma or 'electrons' with mass m and 'ions' with mass M, present in equal density n_0. Then the appropriate $G(v)$, according to (3.10), is:

$$G(v) = \frac{1}{n_0}\left[\frac{\partial f_{0e}}{\partial v} + \frac{m}{M}\frac{\partial f_{0i}}{\partial v}\right]. \tag{9.8}$$

For a plasma of two groups of electrons, $M = m$. In (9.2) the plasma frequency $\omega_p{}^2$ is now the electronic plasma frequency $\omega_{pe}{}^2$. The electrons and ions will be assumed to have Maxwellian velocity distributions with r.m.s. thermal velocities (in one dimension) c_e and c_i and mean velocities u_e and u_i, respectively. Then $G(v)$ becomes

$$G(v) = \frac{-1}{\sqrt{2\pi}c_e{}^2}\left[\left(\frac{v-u_e}{c_e}\right)\exp\left(-\frac{(v-u_e)^2}{2c_e{}^2}\right) + \frac{\theta_e}{\theta_i}\left(\frac{v-u_i}{c_i}\right)\exp\left(-\frac{(v-u_i)^2}{2c_i{}^2}\right)\right] \tag{9.9}$$

* Since Δ is equivalent to the thermal velocity, ω_p/Δ is the effective Debye wave number for this distribution function.

where $\theta_e = mc_e^2$, $\theta_i = Mc_i^2$ are the kinetic temperatures.

Again our aim is to find the minimum velocity difference $V = (u_e - u_i)$ such that growing waves can occur, and to map out the range of k values with growing oscillations as a function of V. With the new variables

$$w_1 = \frac{u_e - v}{c_e}, \qquad w_2 = \frac{v - u_i}{c_i} \tag{9.10}$$

$G(v)$ becomes

$$G(w_1, w_2) = \frac{1}{\sqrt{2\pi}c_e^2}\left[w_1 \exp\left(\frac{-w_1^2}{2}\right) - \frac{\theta_e}{\theta_i} w_2 \exp\left(\frac{-w_2^2}{2}\right)\right] \tag{9.11}$$

with the relation between variables:

$$V = c_e w_1 + c_i w_2 = c_e\left(w_1 + \sqrt{\frac{m\theta_i}{M\theta_e}}\, w_2\right) \tag{9.12}$$

For the case of equal temperatures, the condition $G(x) = 0$ in (9.2) is clearly satisfied by $w_1 = w_2 = w$.* Then the velocity difference is $V = (c_e + c_i)w$. With (9.11) for $G(v)$ the integral in (9.2) can be put in the form:

$$K^2 = -\left[\frac{1}{\sqrt{2\pi}} P\int_{-\infty}^{\infty} \frac{t \exp(-t^2/2)\, dt}{t - w_1} + \frac{\theta_e}{\theta_i}\frac{1}{\sqrt{2\pi}} P\int_{-\infty}^{\infty} \frac{t \exp(-t^2/2)}{t - w_2}\, dt\right] \tag{9.13}$$

where $K^2 = k^2 c_e^2/\omega_{pe}^2$, and w_1, w_2 are evaluated at $v = x$. The integrals in (9.13) are similar to the one evaluated in Appendix 2.† Equation (9.13) can consequently be written:

$$\left(1 + \frac{\theta_e}{\theta_i}\right) + K^2 = w_1^2\, Y\left(\frac{w_1}{\sqrt{2}}\right) + \frac{\theta_e}{\theta_i} w_2^2\, Y\left(\frac{w_2}{\sqrt{2}}\right) \tag{9.14}$$

where the function $Y(x)$ is defined by

$$Y(x) = \frac{\exp(-x^2)}{x}\int_0^x \exp(t^2)\, dt. \tag{9.15}$$

The properties of $Y(x)$ are discussed in Appendix 3. For equal temperatures and $w_1 = w_2 = w$, equation (9.14) can be solved for w as a function of K^2. With (9.12) the range of wave number for which growing waves exist can be found for any velocity difference V.

* There are other solutions to $G(x) = 0$ besides $w_1 = w_2$, but these are of no importance in the present problem.

† The difference is that the variable z in Appendix 2 is now real and the integrals are principal value integrals. The only change is that the imaginary part of (A2.7) for $J(z)$ should be omitted.

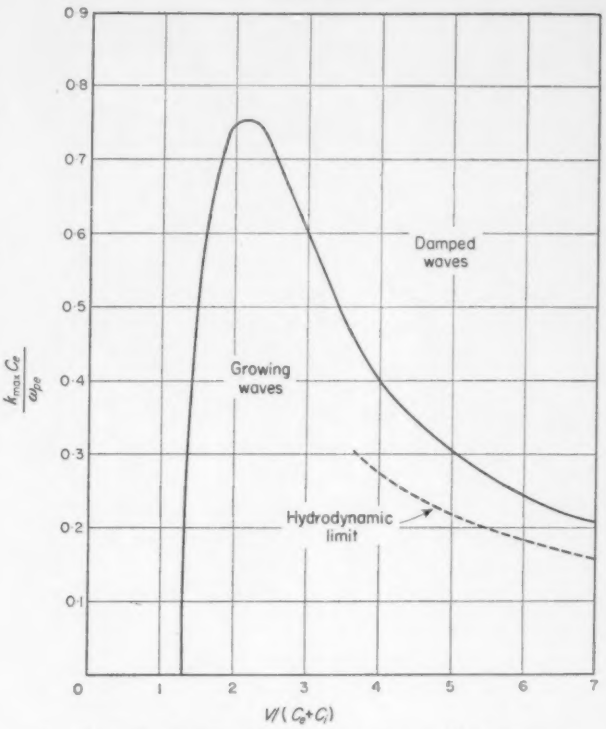

FIG. 5(e).—Same as Fig. 5(d), but for an electron-'ion' plasma with mean velocity difference V and Maxwellian velocity distributions at equal temperature. The velocity difference is shown in units of the sum of the electron and 'ion' r.m.s. velocities $(c_e + c_i)$. The dotted curve represents the hydrodynamic value of k_{max} (9.18) for an electron–proton plasma.

The numerical solution for k_{max} versus V is shown in Fig. 5(e). The minimum velocity difference for the existence of growing waves is $V = 1{\cdot}308(c_e + c_i)$. For an electron-proton plasma this is $1{\cdot}34c_e$, while for two electron groups it is $2{\cdot}62c_e$.‡ The largest range of wave numbers occurs at $V = 2{\cdot}155(c_e + c_i)$, where $k_{max} = 0{\cdot}755\, \omega_{pe}/c_e$. For very large velocity differences, the maximum wave number for which growing waves occur is:

$$k_{max} \simeq \sqrt{2}\left(1 + \sqrt{\frac{m}{M}}\right)\frac{\omega_{pe}}{V}. \tag{9.16}$$

For $m = M$ this reduces to the result found for the two electron groups (9.5).

BUNEMAN (1958) has discussed the dispersion equation for an electron-ion plasma in the hydrodynamic approximation, and has found the complex roots for ω as a function of k. His dispersion equation can be written in terms of our variables as:

$$\frac{1}{w_1^2} + \frac{1}{w_2^2} = K^2 \tag{9.17}$$

‡ This value can be compared to the result $2{\cdot}0\ \Delta$ for the distribution (9.5).

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where w_1 and w_2 (defined by (9.10)) are in general complex.* Equation (9.17) is a quartic equation which has either 4 real roots or 2 real and 2 complex roots. The boundary between these two sets of roots defines the wave number $k_{\max}$ for a given velocity V. It is just this result that was found in equation (7.15). For the extreme cases $m = M$ and $m \ll M$, (7.15) yields the following hydrodynamic values for $k_{\max}$ as a function of V:

$$k_{\max}^{(H)} = \begin{cases} 2\sqrt{2}\,\dfrac{\omega_{pe}}{V} & (m = M) \\ \left(1 + \dfrac{3}{2}\left(\dfrac{m}{M}\right)^{1/3}\right)\dfrac{\omega_{pe}}{V} & (m \ll M) \end{cases} \tag{9.18}$$

For $m = M$, the hydrodynamic result agrees with (9.16); but for $m \ll M$, it is roughly $1/\sqrt{2}$ times the value given by (9.16). For the electron-proton plasma, BUNEMAN's limit (9.18) is plotted in Fig. 5(e).

The discrepancy between (9.18) and (9.16) can be traced to the absence of the requirement $G(x) = 0$ in the hydrodynamic approximation. With delta function velocity distributions $G(x)$ is automatically zero almost everywhere. The hydrodynamic upper limit on wave number can be shown to be less than or equal to the 'true' value (9.16). The case of equality occurs for $m = M$, where $G(x)$ happens to vanish at the appropriate solution of (9.17). In the range of wave numbers between $k_{\max}^{(H)}$ given by (9.18) and $k_{\max}$ given by (9.16), the frequency ω will be almost real since $G(x)$ vanishes in the hydrodynamic approximation and is exponentially small for Maxwellian distributions with a large velocity separation. Consequently, for problems in which the velocity spreads are small compared to the separation of the peaks, the hydrodynamic limit (9.18) is generally a more useful criterion for $k_{\max}$ than (9.16). For small velocity separations, however, the more general result [(9.14) or its equivalent] must be employed.

To illustrate the transition from the region where thermal effects are crucial to the region of the hydrodynamic approximation we consider in detail the solution of the dispersion equation for a plasma of electrons and protons at equal temperatures. The ions are assumed to have zero mean velocity and the electrons a mean velocity V. The dispersion equation is then just the appropriate generalization of (9.13) and can be solved numerically with the help of the tables of FADDEEVA and TERENT'EV (1954) or suitable asymptotic expansions. The real and imaginary parts of the frequency for the root corresponding to growing waves are plotted in Figs. 5(f) and 5(g) for two values of the ratio of the relative velocity to electronic thermal velocity ($V^2/c_e^2 = 10, 30$). For comparison the hydrodynamic approximation ($V^2/c_e^2 \to \infty$) is also shown. Two points are worth mentioning. One is that, while the relatively hot plasma ($V^2/c_e^2 = 10$) gives results appreciably different from the hydrodynamic approximation, for $V^2/c_e^2 \geqslant 30$ the thermal effects consist chiefly in displacing the curves towards slightly higher wave numbers without change in shape. The second point is that in the region of wave number between the hydrodynamic upper limit (9.18) and the 'true' upper limit (9.16), indicated by the vertical arrow, the imaginary part of the frequency is very small for $V^2/c_e^2 \geqslant 30$. This verifies explicitly the general statement of the preceding paragraph.

In terms of the two physical mechanisms for

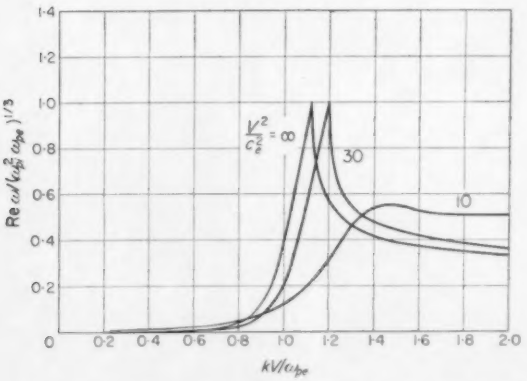

FIG. 5(f).—Real part of the frequency in units of $(\omega_{pi}^2\,\omega_{pe})^{1/3}$ as a function of wave number in units of (ω_{pe}/V) for an electron-proton plasma with a mean velocity difference between components of V and equal temperatures. The three curves correspond to the different values of V^2/c_e^2 indicated.

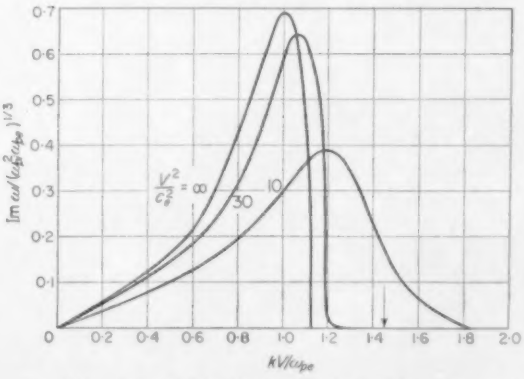

FIG. 5(g).—Same as Fig. 5(f), but for the imaginary part of the frequency. The vertical arrow indicates the 'true' value of $k_{\max}$ (9.16), while the upper end of the $V^2/c_e^2 = \infty$ curve is at the hydrodynamic value (9.18).

* Equation (9.17) can be obtained from the complete dispersion relation (essentially (9.13)) by expanding the functions in powers of ω^{-2}, retaining the lowest order terms, and neglecting all exponentials.

growing oscillations, bunching is dominant for relatively cold plasmas. Only in the upper region of wave numbers does trapping enter, giving the very small contribution to Im (ω) in the hydrodynamically forbidden band. For hot plasmas the distinction between the two mechanisms disappears because the spread in thermal velocities is comparable to the mean velocity difference of the components and the possible phase velocities of the oscillations.

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APPENDIX 1

The analytic function $I(z)$ of the complex variable z is defined by the integral along the real v-axis:

$$I(z) = \int_{-\infty}^{\infty} \frac{f(v)}{v - z}\, dv \tag{A1.1}$$

where $f(v)$ is a real function of v such that the integral exists for finite z. As z approaches the real axis from above or below the Dirac relation (4.5) allows us to express $I(z)$ as follows:

$$\lim_{\mathrm{Im}\,(z)\to 0^{\pm}} I(z) = P\int_{-\infty}^{\infty} \frac{f(v)}{v - x}\, dv \pm \pi i f(x) \tag{A1.2}$$

where $x = \mathrm{Re}\,(z)$.

We wish to generalize the limiting case (A1.2) to obtain a series expansion in powers of $y = \mathrm{Im}\,(z)$. For definiteness we assume $y > 0$. First we consider the expansion of $I(z)$ around the point $z = z_0$:

$$I(z) = I(z_0) + I'(z_0)(z - z_0) + \frac{1}{2} I''(z_0)(z - z_0)^2 + \ldots \tag{A1.3}$$

The derivative of $I(z)$ is

$$I'(z_0) = \int_{-\infty}^{\infty} \frac{f(v)}{(v - z_0)^2}\, dv. \tag{A1.4}$$

Integration by parts transforms (A1.4) into the result:

$$I'(z_0) = \int_{-\infty}^{\infty} \frac{f'(v)}{v - z_0}\, dv. \tag{A1.5}$$

Similarly the nth derivative $I^{(n)}(z_0)$ can be expressed in terms of the nth derivative of $f(v)$:

$$I^{(n)}(z_0) = \int_{-\infty}^{\infty} \frac{f^{(n)}(v)}{v - z_0}\, dv. \tag{A1.6}$$

If we now consider z_0 approaching the real axis from above the Dirac expression (A1.2) can be used to write (A1.6):

$$\lim_{\mathrm{Im}\,(z_0)\to 0^{+}} I^{(n)}(z_0) = P\int_{-\infty}^{\infty} \frac{f^{(n)}(v)}{v - x}\, dv + \pi i f^{(n)}(x). \tag{A1.7}$$

Consequently $I(x + iy)$ can be expanded using (A1.3) and (A1.7) as follows:

$$I(x + iy) = \sum_{n=0}^{\infty} \frac{1}{n!} (iy)^n \left[P\int_{-\infty}^{\infty} \frac{f^{(n)}(v)}{v - x}\, dv + \pi i f^{(n)}(x) \right] \tag{A1.8}$$

The first term in (A1.8) is just the Dirac result (A1.2). Further terms give corrections for non-zero values of y. In the lower half-plane we write $z = x - iy$. Then the proper expansion is obtained by taking the complex conjugate of (A1.8). One virtue of the expansion (A1.8) is that the real and imaginary parts of $I(z)$ can be exhibited explicitly.

APPENDIX 2

With the Maxwellian $G(v)$ given by (6.1) the dispersion equation (4.4) is:

$$0 = K^2 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{v \exp(-v^2/2)}{v - z}\, dv + \sqrt{2\pi}\, iz \exp(-z^2/2) \tag{A2.1}$$

where $z = \omega/kc$, $K = kc/\omega_p$, and it is assumed that z has a negative imaginary part. With a change of variable $t = v - z$, the integral can be written:

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{v \exp(-v^2/2)}{v - z}\, dv$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{t + z}{t}\right) \exp\{-(t + z)^2/2\}\, dt. \tag{A2.2}$$

Then the first term in (A2.2) integrates to unity, and I becomes:

$$I = 1 + z \exp(-z^2/2)\, J(z) \tag{A2.3}$$

where

$$J(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{t} \exp\{-(t^2/2 + zt)\}\, dt. \tag{A2.4}$$

To evaluate $J(z)$ we note that

$$\frac{dJ(z)}{dz} = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-(t^2/2 + zt)\}\, dt = -\exp(+ z^2/2). \tag{A2.5}$$

Also

$$J(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{t} \exp(-t^2/2)\, dt. \tag{A2.6}$$

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In this last integral the path of integration is just above the real axis, as can be checked from the change of variable $t = v - z$. With the Dirac definition (4.5), $J(0)$ can be written as a principal value integral minus (πi) times the integrand evaluated at $t = 0$. The principal value integral vanishes, so that

$$J(0) = -\mathrm{i}\sqrt{\frac{\pi}{2}}.$$

Then $J(z)$ can be written as

$$J(z) = -\mathrm{i}\sqrt{\frac{\pi}{2}} - \int_0^z \exp(y^2/2)\,dy. \qquad \text{(A2.7)}$$

The dispersion equation (A2.1) is therefore

$$0 = K^2 + 1 + \sqrt{\frac{\pi}{2}}\, z \exp(-z^2/2)\left(\mathrm{i} - \frac{2}{\sqrt{\pi}}\int_0^{z/\sqrt{2}} \exp(t^2)\,dt\right). \qquad \text{(A2.8)}$$

The remaining integral is readily identified as the error function of complex argument. The tables of FADDEEVA and TERENT'EV (1954) tabulate the function $W(z)$ defined as the Hilbert transform of a Gaussian:

$$W(z) \equiv \frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\exp(-u^2)}{u - z}\,du = \exp(-z^2) \times \left(1 + \frac{2\mathrm{i}}{\sqrt{\pi}}\int_0^z \exp(t^2)\,dt\right). \qquad \text{(A2.9)}$$

The range of z covered is $x, y > 0$, $x, y = 0(0{\cdot}02)3$ and $x, y = 3(0{\cdot}10)5$, with the real and imaginary parts of $W(z)$ tabulated to six places. In terms of $W(z)$ the dispersion equation (A2.8) becomes

$$0 = K^2 + 1 + \mathrm{i}\sqrt{\frac{\pi}{2}}\, z\, W(z/\sqrt{2}). \qquad \text{(A2.10)}$$

The numerical solution can be found using the tables by first determining the value of x for which the imaginary part of (A2.10) is satisfied for a chosen value of y, and then solving the real part of (A2.10) for the corresponding value of K^2. The problem of needing $W(z)$ for negative y while the tables are for positive y can be handled by noting that if (x, y) satisfy the dispersion relation then $(-x, y)$ is also a solution (corresponding to a wave moving in the opposite direction). Then the relation

$$W(-z) = 2\exp(-z^2) - W(z) \qquad \text{(A2.11)}$$

allows one to obtain $W(z)$ for the required negative y values. The numerical results for the solution of (A2.10) are shown in Fig. 3.

A remark needs to be made about other roots of the dispersion relation besides the one shown in Fig. 3. To show that such roots exist it is sufficient to examine the form of $W(z)$ for large values of $z(y > 0)$. With the integral definition (A2.9) it is easy to obtain the asymptotic expansion:

$$W(z) \to \frac{\mathrm{i}}{\sqrt{\pi}z}\left(1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \ldots + \frac{(2n-1)!!}{2^n z^{2n}} + \ldots\right). \qquad \text{(A2.12)}$$

The presence of the exponential in (A2.11) together with the form (A2.12) implies oscillatory factors in the dispersion relation (A2.10) and the presence of an infinite number of roots. Only the root with smallest imaginary part is of interest since it is on the branch which reduces to the plasma frequency (1.1) and Landau damping constant (1.2) at vanishing wave number. Other roots lie on branches which have damping constants that do not vanish even at $k = 0$. These are of little importance since they are presumably damped out rapidly compared to the characteristic time ω_p^{-1}.

APPENDIX 3

The function $Y(x)$ defined by

$$Y(x) = \frac{1}{x}\exp(-x^2)\int_0^x \exp(t^2)\,dt \qquad \text{(A3.1)}$$

is related to the error function of imaginary argument and to the imaginary part of $W(z)$ (A2.9) for real z. Four-figure tables

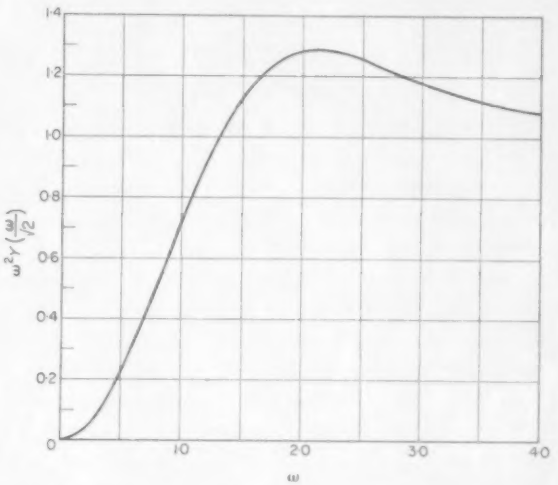

FIG. 6.—The function $w^2 Y(w/\sqrt{2})$ plotted as a function of w.

of $Y(x)$ have been computed at Space Technology Laboratories, Inc. for $x = 0(0{\cdot}025)10$*. In Section 9 the function $w^2 Y\left(\frac{w}{\sqrt{2}}\right)$ is employed. Fig. 6 shows a plot of $w^2 Y\left(\frac{w}{\sqrt{2}}\right)$ for the range $0 \leqslant w < 4$. Some useful expansions are:

Small w

$$w^2 Y\left(\frac{w}{\sqrt{2}}\right) = w^2\left[1 - \frac{w^2}{3} + \frac{w^4}{15} - \ldots \frac{(-1)^n w^{2n}}{(2n+1)!!} + \ldots\right]. \qquad \text{(A3.2)}$$

Large w

$$w^2 Y\left(\frac{w}{\sqrt{2}}\right) = \left[1 + \frac{1}{w^2} + \frac{3}{w^4} + \frac{15}{w^6} + \ldots + \frac{(2n-1)!!}{w^{2n}} + \ldots\right]. \qquad \text{(A3.3)}$$

Expansion around $w_0 = 1{\cdot}308$

$$w^2 Y\left(\frac{w}{\sqrt{2}}\right) = \left[1 + \frac{1}{w_0}(w - w_0) - \frac{1}{2}(w - w_0)^2 + \ldots\right]. \qquad \text{(A3.4)}$$

* A short table of $x\exp(x^2)Y(x)$ is given by JAHNKE F. and EMDE E. (1945) *Tables of Functions* (4th ed.) p. 32. Dover Publ., New York.

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LONGITUDINAL PLASMA OSCILLATIONS IN AN ELECTRIC FIELD

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Abstract—The properties of longitudinal plasma oscillations in an external electric field are investigated. In a completely linear approximation, it is found that the direct-current electric field introduces essentially no new effects. A quasi-linear approximation is also considered, in which couplings between different plasma modes are neglected while the space-averaged distribution functions are assumed to be approximately independent of time. In this case, a Maxwellian distribution function is found to be always unstable against the growth of very long wavelength oscillations.

1. INTRODUCTION

IN the course of an attempt to understand in more detail the possibility, suggested by BUNEMAN (1958), that long-range co-operative effects in the form of growing plasma waves may provide a new mechanism for plasma resistivity, we have studied the dispersion equation for longitudinal plasma waves in the presence of an external electric field. While we have not, as yet, succeeded in achieving a quantitative understanding of BUNEMAN'S mechanism, the results concerning the effect of an electric field on plasma waves are self-contained and may be of value also in other investigations.

We consider a plasma composed of electrons and ions and assume that the distribution function (in phase space) for each species obeys a collisionless Boltzmann equation, with electromagnetic fields whose sources are the plasma charge and current density. Since the two-stream instability which BUNEMAN considers involves only longitudinal plasma waves, we neglect the magnetic field due to the plasma current. We also assume that no external magnetic field is present. The problem is then essentially one-dimensional, and we have for the electron distribution function, $f(x, v, t)$,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}(E_e + E)\frac{\partial f}{\partial v} = 0. \qquad (1)$$

The ion distribution function, F, satisfies the same equation with $e/m \to -e/M$. The external electrical field $E_e(t)$ is a given function of time, while the self-consistent plasma field, E, is determined from Poisson's equation

$$\frac{\partial E}{\partial x} = 4\pi e \int dv(F - f). \qquad (2)$$

If $E_e = 0$, the linearized form of these equations can readily be solved. The resulting dispersion equation (JACKSON, 1958) predicts Landau damping (LANDAU, 1946) if the unperturbed distributions have no relative mean velocity and gives growing waves if the mean velocities differ by more than ε times the electron thermal velocity (for $T_i = T_e$), where ε is a number of order one whose exact value (JACKSON, 1958) depends upon the form assumed for the unperturbed velocity distributions. It is the aim of the present paper to generalize these field-free results and to examine the effect of an external electric field upon the plasma waves.

With the usual separation of f into a space averaged part, f_0, and the fluctuations, f_1, around that, we find that in a strictly linear theory f_0 must be time-dependent. Consequently, the equation for f_1 does not have harmonic solutions and there is no dispersion equation in the usual sense. This is discussed in Section 2. In Section 3 we consider briefly the consequences of assuming f_0 to be time-independent, as might be appropriate in a quasi-linear theory which takes account of the effects of the fluctuations upon f_0 but neglects the coupling among the fluctuation modes. In this case a dispersion equation of the usual sort can be derived and leads to growing waves with a Maxwellian f_0 even in absence of a relative electron-ion drift. We conclude that either the quasi-linear approximation with time-independent f_0 is inherently inconsistent or else that it demands a special form for f_0, different in character from a Gaussian.

2. THE LINEAR THEORY

It is convenient to make a Fourier expansion of the x dependence of the distribution functions,

$$f(x, v, t) = n_0 f_0(v, t) + \sum_k f_k(v, t) \exp ikx$$

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with similar expansions for F and for

$$E(x, t) = \sum_k E_k(t) \exp(ikx).$$

Reality of f,E requires $f_k^* = f_{-k}$, etc. The space averaged density of both ions and electrons is indicated by n_0, and f_0 is normalized to 1. (The k spectrum is made discrete by using periodic boundary conditions with a period L so that the allowed k values are multiples of $2\pi/L$.) The equations for the Fourier amplitudes are then

$$\frac{\partial f_0}{\partial t} - \frac{eE_e}{m}\frac{\partial f_0}{\partial v} = \frac{e}{mn_0}\Sigma E_k^* \frac{\partial f_k}{\partial v} \tag{3}$$

$$\frac{\partial f_k}{\partial t} + ikvf_k - \frac{eE_e}{m}\frac{\partial f_k}{\partial v} - \frac{n_0 e}{m}E_k\frac{\partial f_0}{\partial v} = \frac{e}{m}\sum_{k'} E_{k'+k}\frac{\partial f_{k'}^*}{\partial v} \tag{4}$$

$$ikE_k = 4\pi e\int dv(F_k - f_k). \tag{5}$$

In the linearized approximation we drop the right-hand sides of equations (3) and (4). Then equation (3) is solved by taking f_0 to be an arbitrary function of

$$u = v + (e/m)\int_0^t E_e(t') \tag{6}$$

Introducing u and t as independent variables in place of v and t, we can write equation (4) as

$$\frac{\partial f_k}{\partial t} + ik\left[u - \frac{e}{m}\int_0^t E_e(t')\,dt'\right]f_k = \frac{n_0 e}{m}E_k\frac{df_0}{du} \tag{7}$$

with a similar equation for F_k.

Since the coefficients are time-dependent, the solutions of equation (7) are not plane waves and we cannot find a dispersion equation in the usual sense. However, we can solve equation (7) by using an integrating factor,

$$f_k(u, t) = \exp\{-ik[ut - \phi(t)]\}\left\{n_0\frac{e}{m} \times \int_0^t dt' E_k(t')\exp\{ik[ut' - \phi(t')]\}\frac{df_0}{du} + f_k(u, 0)\right\} \tag{8}$$

where

$$\phi(t) = (e/m)\int_0^t dt'(t - t')E_e(t').$$

The electron density is then

$$n_k(t) = \int_{-\infty}^{\infty} du f_k(u, t) = \exp\{ik\phi(t)\}\left\{ik\frac{n_0 e}{m} \times \int_0^t dt' E_k(t')(t - t')\,.\,\bar{f}_0[k(t - t')] \times \exp\{-ik\phi(t')\} + \bar{f}_k[kt, 0]\right\} \tag{9}$$

where the bar denotes a Fourier transform with respect to u,

$$\bar{f}_0(0) = \int_{-\infty}^{\infty} du \exp\{-iu\theta\} f_0(u) \tag{10}$$

$$\bar{f}_k(\theta, 0) = \int_{-\infty}^{\infty} du \exp\{-iu\theta\} f_k(u, 0),$$

and an integration by parts has been used to transfer (d/du) from f_0 to $\exp[iku(t' - t)]$. Substituting equation (9) and a similar expression for ion density into Poisson's equation, equation (5), we obtain finally an integral equation for $E_k(t)$.

$$E_k(t) + \omega_p^2\int_0^t dt' E_k(t')(t - t') \left\{\bar{f}_0[k(t - t')] \times \exp[ik(\phi - \phi')] + \frac{m}{M}\bar{F}_0[k(t - t')] \times \exp[-ikm(\phi - \phi')/M]\right\} = (4\pi ei/k)[\exp(ik\phi)\bar{f}_k(kt, 0) - \exp(-ikm\phi/M)\,\bar{F}_k(kt, 0)] \tag{11}$$

where $\omega_p^2 = 4\pi n_0 e^2/m$ is the electron plasma frequency.

In absence of the external electric field, $\phi = 0$ and the integral equation is of the convolution type. A solution is readily obtained by means of Laplace or one-sided Fourier transforms,

$$E_k(\omega) = \frac{R(\omega)}{1 + D(\omega)} \tag{12}$$

where $R(\omega)$ is the transform of the right-hand side of equation (11) and $D(\omega)$ is the transform of

$$\omega_p^2 t[\bar{f}_0(kt) + (m/M)\bar{F}_0(kt)].$$

The necessary and sufficient condition for stability of the oscillations is that the denominator of equation (12) have no roots in the upper half-ω-plane. This problem and the properties of $D(\omega)$ have been carefully discussed by JACKSON (1958).

The integral equation is also simple if only electron fluctuations are considered. In the limit $m/M \to 0$ we have again a convolution equation, this time for the quantity $E\exp(-ik\phi)$. Since ϕ is real, the stability properties are identical with those in absence of an external field.

For the case where neither m/M nor E_e vanishes, equation (11) is rather formidable. For any given initial conditions, the right-hand side of equation (11) is known and one could at least obtain a numerical solution. To determine the stability properties, however, it is necessary to decide whether equation

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(11) has solutions with unbounded E for *any* initial conditions. This information is readily obtained from the usual dispersion equation but we do not know a general technique for extracting it from the integral equation. Some progress can be made by rewriting equation (11) in terms of a formal operator representation, as follows. We solve equation (7) by formally inverting the differential operator,

$$f_k = \frac{n_0 e}{m}\left[\frac{\partial}{\partial t} + ik(u - \dot{\phi})\right]^{-1} E_k \frac{df_0}{du}. \tag{13}$$

The density is then

$$n_k = \int du f_k = \frac{n_0 e}{mik}\, g\left[\frac{i}{k}\frac{\partial}{\partial t} + \dot{\phi}\right] E_k \tag{14}$$

where the function g is defined by

$$g(x) = \int_{-\infty}^{\infty} du(u - x - i\varepsilon)^{-1}\frac{df_0}{du}, \tag{15}$$

the singularity in the integrand for real x being defined in the manner appropriate to an initial-value problem. Substitution of equation (14) and the analogous expression for ion density into Poisson's equation gives the operator form of equation (11),

$$E_k = \left(\frac{\omega_p^2}{k^2}\right)\left[g\left(\frac{1}{k}\frac{\partial}{\partial t} + \dot{\phi}\right) + \left(\frac{m}{M}\right) G\left(\frac{1}{k}\frac{\partial}{\partial t} - \frac{m}{M}\dot{\phi}\right)\right] E_k \tag{16}$$

where G is defined as in equation (15) with the ion distribution, F_0, in place of f_0.

If f_0 is Maxwellian,

$$f_0(u) = \frac{\exp(-u^2/a^2)}{\pi^{1/2}a} \tag{17}$$

then

$$g(\xi) = a^{-2}Z'(\xi/a)$$

where $Z(x)$ is the 'plasma dispersion function' which is always encountered in an analysis of plasma oscillations linearized about a Maxwellian distribution

$$Z(x) \equiv \pi^{-1/2}\int_{-\infty}^{\infty} d\theta(\theta - x - i\varepsilon)^{-1}\exp(-\theta^2)$$
$$= 2i\exp(-x^2)\int_{-\infty}^{ix} \exp(-q^2)\, dq$$
$$= i\sqrt{\pi}\exp(-x^2) - 2xY(x),$$

$Y(x)$ being real for real x,

$$Y(x) \equiv \exp(-x^2)x^{-1}\int_0^x \exp(q^2)\, dq. \tag{18}$$

(For some useful properties of Y and Z, see JACKSON (1958)). Even in the low-temperature limit ($a \to 0$) equation (16) is complicated, for the asymptotic form of Z' is (see Appendix)

$$Z'(x) \to -2\sigma i\sqrt{\pi}x\exp(-x^2) + x^{-2} \quad \text{for } |x| \to \infty, \tag{19}$$

where $\sigma = 0$, 1, or 2 according to whether $\mathrm{Im}(x)$ is positive, zero, or negative. For the corresponding g it follows that

$$g(\xi) \to \xi^{-2} - 2i\sigma\sqrt{\pi}(\xi/a^3)\exp(-\xi^2/a^2) \quad \text{for } |\xi| \to \infty. \tag{20}$$

Instead, we shall use the simpler function,

$$g(\xi) = (\xi + ia)^{-2} \tag{21}$$

which corresponds to the choice of a resonance shape distribution function

$$f_0(u) = \pi^{-1}a(u^2 + a^2)^{-1}. \tag{22}$$

For the case where the two species have equal velocity spreads and equal masses ($m = M$, $a = A$), the equation for E_k is then

$$E_k = \frac{\omega_p^2}{k^2}\left[\frac{1}{\left(\frac{i}{k}\frac{\partial}{\partial t} + \dot{\phi} + ia\right)^2} + \frac{1}{\left(\frac{i}{k}\frac{\partial}{\partial t} - \dot{\phi} + ia\right)^2}\right] E_k \tag{23}$$

The ia term in the denominators, which represents Landau damping for our particular f_0, can be eliminated by the substitution

$$E_k(t) = \exp(-akt)y(t). \tag{24}$$

Then equation (23) becomes

$$\frac{k^2}{\omega_p^2}\, y = (\alpha^{-2} + \beta^{-2})y \tag{25}$$

where

$$\alpha = \frac{i}{k}\frac{\partial}{\partial t} + \dot{\phi} \qquad \beta = \frac{i}{k}\frac{\partial}{\partial t} - \dot{\phi}. \tag{26}$$

Rationalizing the denominators in equation (25) and setting

$$y = \beta^2\eta \tag{27}$$

we have finally a fourth-order equation for η,

$$(\alpha^2 + \beta^2)\eta = (k^2/\omega_p^2)a^2\beta^2\eta \tag{28}$$

We now specialize to the case of a constant external field. Since electrostatic instabilities tend to be more serious for the longer wavelengths, we first study equation (28) in the limit of very small k. An explicit definition of the 'small k' regime can be obtained by imagining that the external field is switched off at

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time t, leaving the two species with velocities $\pm V = \pm eE_e t/m$. The differential equation (28) can then be solved with an exponential $\exp(iukt)$, where u is the root of

$$2(u^2 + V^2) = (k^2/\omega_p^2)(u^2 - V^2)^2.$$

The correction to the $k = 0$ solution, $u^2 = -V^2$, is small provided $kV/\omega_p \ll 1$. Thus, we consider k as 'small' if

$$k \ll m\omega_p/eE_e t. \tag{29}$$

If we define

$$s = kt \quad \text{and} \quad \bar{\gamma} = eE_e/km$$

then the only explicit occurrence of k is in the factor k^2 on the right side of equation (28). In the limit of small k we then have

$$(\alpha^2 + \beta^2)\eta = 2(\bar{\gamma}^2 s^2 - \partial^2/\partial s^2)\eta = 0$$

whose general solution is

$$\eta = s^{1/2} Z_{1/4}(i\bar{\gamma}s^2/2) \tag{30}$$

where $Z_{1/4}$ denotes any Bessel function of order 1/4.

The character of the small k solution is now clear. For some choice of initial conditions, the Bessel function in equation (30) will involve at least some of the Hankel function of second kind, so that $\eta(s)$ will grow exponentially

$$\eta(s) \approx \exp(\bar{\gamma}s^2/2) \tag{31}$$

for

$$\bar{\gamma}s^2/2 \gg 1.$$

It follows from equations (27) and (24) that y will have the same growth character as η, while E_k will grow only when the increasing exponential in equation (31) exceeds the Landau damping, i.e.

$$\bar{\gamma}s^2/2 > as. \tag{32}$$

These results can most conveniently be summarized in terms of three characteristic times:

$$T = m\omega_p/eE_e k,$$

the time for the field to produce particle velocities of ω_p/k;

$$t_g = \sqrt{T/\omega_p} = \sqrt{m/eE_e k},$$

the time at which the Hankel function begins its exponential growth (corresponding to $\bar{\gamma}s^2 = 1$);

$$t_d = 2ma/eE_e = 2(ka/\omega_p)T,$$

the time at which equation (32) is satisfied and also the time required for the field to produce a relative drift velocity of order a.

For given k, it follows from equation (29) that the solution of equation (30) is valid only for $t < T$. Thus, there are three possibilities.

(a) If the values of a and E_e are such that

$$t_g < t_d < T$$

or

$$eE_e/kma^2 < 1 < (k_D/2k)^2 \tag{33}$$

(where $k_D = \omega_p/a$ is the Debye wave number), then the Hankel function growth starts at a time (t_g) when its rate is less than the Landau damping. Later on, (at t_d) but still before $t = T$, the relative drift velocity exceeds a and E_k begins an exponential growth which continues at least until time T.

(b) If

$$t_d < t_g < T, \quad \text{or} \quad 1 < eE/mka^2 < (k_D/k)^2 \tag{34}$$

then even though the relative drift velocity exceeds a at time t_d, growth of E is postponed until the later time (t_g) when the Hankel function attains its asymptotic character. This result is at first surprising; in the case $E_e = 0$ a drift velocity greater than a leads to growth, so that one would here expect growth at a time of order t_d. However, the energy exchange between particles and wave which constitutes the physical reason for growth of the wave (JACKSON, 1958) cannot occur in a time less than that required for a particle to traverse one wavelength, and this time is just t_g.* Hence we have the double condition for growth in presence of an electric field: t must be great enough for the external field to produce a relative drift velocity greater than the thermal speed and also to accelerate the particles through a distance of at least one wavelength.

(c) If

$$t_g > T \quad \text{or} \quad t_D > T$$

that is, if

$$k > k_D \text{ or } eE/mka^2 > (k_D/k)^2 \tag{35}$$

then we can only conclude that no growth of E_k occurs before a time T. Whether it occurs subsequently can only be determined by dropping the restriction to small k or small t.

In the opposite limit of large k or large t, we expect that an approximate solution should follow from setting the right-hand side of equation (28) equal to zero. Noting equation (27) we then have

$$\alpha^2 y = 0 \tag{36}$$

whose general solution is $y = (c_1 s + c_2)\exp(-i\gamma s^2/2)$ where c_1 and c_2 are constants. Thus, y has no exponential growth and the Landau damping, $\exp(-as)$, prevails. The physical reason for the absence of growth is simply that at times greater than T the

* The time for a particle to go a distance $1/k$ in virtue of its thermal velocity alone is greater than t_g when the inequality equation (34) holds.

electric field has accelerated all particles to velocities greater than the phase velocity of plasma waves, ω_p/k, leaving no particles to be trapped by the waves. We see that the general characteristics are just those to be expected from consideration of the field-free case, the only new features being the requirement that growing waves occur only if there is time to accelerate a particle through one wavelength, and that after long times ($t \gg T$) waves of a given k stop growing and decay by Landau damping. It seems reasonable to expect a similar behaviour in the case $m \neq M$ and also for other choices of f_0, but we have not explicitly demonstrated this.

3. A QUASI-LINEAR APPROXIMATION

We now adopt a different point of view. Instead of assuming the fluctuations to have an amplitude small enough to permit complete linearization, we suppose that as a consequence of BUNEMAN'S mechanism a kind of quasi-equilibrium is established in which f_0 and F_0 are nearly time-independent. This can come about only if the amplitudes of the fluctuations have increased to a point where the right-hand side of equation (3) approximately balances the term containing E_e. In fact, we would require f_0 and F_0 to have such shapes as to lead to little growth of the f_k, while also demanding that the f_k have a velocity dependence which enables the nonlinear term in equation (3) to cancel the E_e term. It is far from clear whether the equations have any self-consistent solution of this character. As a first step in studying this, however, we have examined the consequences of assuming that

(a) f_0 is independent of time,

(b) the nonlinear terms in equation (4) can be neglected,

(random phase approximation). At worst, this can be regarded as an approximation to the problem discussed in the previous section, valid over times short compared to that in which f_0 changes appreciably

$$t \ll m \int (dv\, vf_0)/eE_e.$$

Thus, we study the linear system

$$\frac{\partial f_k}{\partial t} + \mathrm{i}kvf_k - \frac{eE_e}{m}\frac{\partial f_k}{\partial v} = \frac{n_0 e}{m} E_k \frac{\partial f_0}{\partial v} \tag{37}$$

with a similar equation for F_k and with

$$E_k = 4\pi e \int dv(F_k - f_k). \tag{38}$$

We shall assume that E_e is independent of time. The general solution of equation (37) (obtained, for example, by straightforward application of the method of characteristics) is then

$$f_k(v, t) = \left(n_0 \frac{e}{m}\right) \int_0^t dt' E_k(t') \left[\frac{df_0}{dv}(v + \lambda\tau, t') \exp\left\{-ik\left(\frac{\lambda\tau^2}{2} + v\tau\right)\right\}\right] + \exp\{-\mathrm{i}k(vt + \lambda t^2/2)\} f_k(v + \lambda t, 0) \tag{39}$$

where

$$\tau = t - t', \quad \lambda = eE_e/m.$$

The electron density is

$$n_k = \int dv\, f_k(v) = \left(n_0 \frac{e}{m}\right) \int_0^t dt'\, E_k(t') \mathrm{i}k\tau \,[\bar{f}_0(k\tau, t') \exp(\mathrm{i}k\lambda\tau^2/2) + \bar{f}_k(kt, 0) \exp(\mathrm{i}k\lambda t^2/2)] \tag{40}$$

where the bar denotes the Fourier transform, defined as in equation (10). Upon substituting this and an analogous expression for ion density into Poisson's equation, equation (5), we obtain again an integral equation for $E_k(t)$,

$$E_k(t) + \omega_p^2 \int_0^t dt' E_k(t')\, \tau\, [\bar{f}_0(k\tau, t') \exp(\mathrm{i}k\lambda\tau^2/2) + \bar{F}_0(k\tau, t') \exp(-\mathrm{i}km\lambda\tau^2/2M] = X(t) \tag{41}$$

where

$$X(t) = (4\pi e/\mathrm{i}k)\,[\bar{F}_k(kt, 0) \exp(-ik\lambda t^2 m/2M) - \bar{f}_k(kt, 0) \exp(\mathrm{i}k\lambda t^2/2)] \tag{42}$$

depends on the initial conditions.

When f_0 is independent of time, the integral equation (41) is of convolution type, and the solution by Laplace transform is immediate. With

$$E_k(\omega) = \int_0^\infty dt E_k(t) \exp(\mathrm{i}\omega t) \tag{43}$$

and a similar definition for $X(\omega)$ we have

$$E_k(\omega) = X(\omega)\,\{1 + \omega_p^2[r(\omega) + R(\omega)]\}^{-1} \tag{44}$$

where r and R are transforms of the kernels of equation (41),

$$r(\omega) = \int_0^\infty dt\, \bar{f}_0(kt)\, t \exp\{\mathrm{i}(\omega t + k\lambda t^2/2)\} = (-\mathrm{i}/k^2) \frac{d}{du} \int_0^\infty d\theta\, \bar{f}_0(\theta) \exp\{\mathrm{i}(u\theta + \lambda\theta^2/2k)\} \tag{45}$$

with

$$u = \omega/k.$$

$R(\omega)$ is defined in an analogous fashion. In inverting equation (43),

$$E_k(t) = \int_{-\infty}^{\infty} d\omega \frac{\exp(-i\omega t)\, X(\omega)}{1 + \omega_p{}^2 [R(\omega) + r(\omega)]} \qquad (46)$$

the integral is to be carried out along a contour which passes above all of the singularities of the integrand. Aside from poles of $X(\omega)$, which depend upon the particular initial conditions chosen, the poles of the integrand will occur at points where the denominator vanishes.

$$D(\omega) = 1 + \omega_p{}^2[r(\omega) + R(\omega)] = 0. \qquad (47)$$

If equation (47), which is just the dispersion relation for this system, has roots in the upper half-plane, then $E_k(t)$ will grow exponentially at large times, i.e. the oscillations will be unstable.

To gain some familiarity with the dispersion equation (47), we investigate its properties for the particular case of Maxwellian distributions for f_0 and F_0. We choose a frame in which the drift velocities are $\pm V$ and we assume both species to have the same temperature,

$$f_0 = \frac{\exp\{-(v - V)^2/a_1{}^2\}}{\sqrt{\pi} a_1}$$

$$F_0 = \frac{\exp\{-(v + V)^2 a_2{}^2\}}{\sqrt{\pi} a_2} \quad a_2{}^2 = (m/M) a_1{}^2. \qquad (48)$$

The Fourier transform of f_0 is

$$\tilde{f}_0(\theta) = \int dv \exp(-iv\theta) f_0(v) = \exp\{-[a_1{}^2\theta^2/4 + iV\theta]\} \qquad (49)$$

and the function r required for the dispersion equation is

$$r(\omega) = \frac{-i}{k^2} \frac{d}{du} \int_0^{\infty} d\theta \tilde{f}_0(\theta) \exp\{i(u\theta + \lambda\theta^2/2k)\}$$

$$= -\frac{1}{k^2} \frac{d}{du} \frac{1}{a_1 \mu_1} Z\left(\frac{u - V}{a_1 \mu_1}\right) \qquad (50)$$

where

$$\mu_1 = \sqrt{(1 - 2i\lambda/k a_1{}^2)}, \quad u = \omega/k$$

and Z is the 'plasma dispersion function' defined in equation (18). (The reduction of the integral in equation (50) to the Z function requires just some completions of the square in the exponent.) The dispersion equation (47) is then

$$1 = (\omega_p{}^2/k^2 a_1{}^2) \left[\frac{1}{\mu_1{}^2} Z'\left(\frac{u - V}{\mu_1 a_1}\right) + \frac{1}{\mu_2{}^2} Z'\left(\frac{u + V}{\mu_2 a_2}\right)\right] \qquad (51)$$

where

$$\mu_2 = \mu_1{}^*.$$

The dimensionless parameter $\sqrt{(2\lambda/ka^2)}$ is just the ratio of the velocity increment produced by the field in a distance $1/k$ to the thermal velocity. In the limit $\lambda = 0$, equation (51) reduces, as it should, to the dispersion relation given by JACKSON (1958). For $\lambda \neq 0$ but $\lambda/ka_1{}^2 \ll 1$ the properties are qualitatively similar to the zero field case. However, for $\lambda/ka_1{}^2 \gg 1$, the character is quite different. In particular, we find that growing waves occur for arbitrarily small values of the drift velocity V, and, in fact, even in the limit $M/m \to \infty$ where the ions are very heavy and do not participate in the oscillations.

Consider the latter case, i.e. an electron plasma with a background of heavy positive ions to provide charge neutrality. We want to know whether the the dispersion equation, which now simplifies to

$$\frac{k^2 a^2}{\omega_p{}^2} - \frac{1}{\mu^2} Z'\left(\frac{u}{\mu a}\right) = 0 \qquad (52)$$

has any roots with $\mathrm{Im}(u) > 0$. The use of a Nyquist diagram, as described by JACKSON (1958), enables us to answer this without the necessity of evaluating equation (52) for complex u. Unfortunately, even if u is real, the argument of Z' is complex because of μ, and the separation of Z into real and imaginary parts is simple only when the argument is real, pure imaginary, or proportional to $\sqrt{i}$. We therefore exploit the fact that in the large field limit, $\lambda/ka^2 \gg 1$, μ^2 is nearly pure imaginary. Introducing the velocity

$$\gamma = \sqrt{\lambda/2k} = \sqrt{eE_e/2mk} \qquad (53)$$

(we shall assume that both k and E_e are positive) we have

$$\mu = \sqrt{-4i\gamma^2/a^2 + 1}$$

$$= 2\gamma/a\sqrt{-i}(1 + ia^2/8\gamma^2 + \ldots). \qquad (54)$$

If we neglect the a^2/γ^2 term, then

$$\frac{1}{\mu^2} Z'\left(\frac{u}{a\mu}\right) = -\frac{a^2}{2\gamma^2}\left\{x\sqrt{\frac{\pi}{2}}\left[\sin x^2 \left(\frac{2x}{|x|} S[x^2] - 1\right)\right.\right.$$

$$\left. + \cos x^2 \left(\frac{2x}{|x|} C[x^2] - 1\right)\right] + i\left[1 + x\sqrt{\frac{\pi}{2}} \cos x^2\right.$$

$$\times \left(\frac{2x}{|x|} S[x^2] - 1\right) - x\sqrt{\frac{\pi}{2}} \sin x^2$$

$$\left.\left. \times \left(\frac{2x}{|x|} C[x^2] - 1\right)\right]\right\}, \; x = \frac{u}{2\gamma} \qquad (55)$$

where C and S are the Fresnel integrals

$$C(x^2) + iS(x^2) \equiv \sqrt{\frac{2}{\pi}} \int_0^x \exp(it^2)\, dt, \quad x > 0. \qquad (56)$$

For small or intermediate values of $x = u/2\gamma$, the representation equation (55) is a good approximation for large γ/a. However, in the asymptotic region ($x \ll -1$) it is not correct; the real part of μ^2 causes a damping of the linearly divergent, oscillatory character predicted by equation (55). To show this, we use the large argument asymptotic form* of Z,

$$Z'(x) = \{-4i\sqrt{\pi}x \exp(-x^2)\} + \frac{1}{x^2} + \frac{3 \cdot 1}{2x^4} + \frac{5 \cdot 3 \cdot 1}{2^2 x^6} + \cdots$$

where the term in brackets is to be included if, and only if, $\mathrm{Im}(x) < 0$. Including the a^2/γ^2 correction to μ in equation (54) we then find

$$\frac{1}{\mu^2} Z'\left(\frac{u}{a\mu}\right) = \frac{a^2}{2\gamma^2}\left\{\frac{u}{\gamma}\eta(-u)\sqrt{\pi}i \exp\left[-\frac{a^2}{\gamma^2}\frac{u^2}{16\gamma^2}\right] \times \exp\{-i(u/2\gamma)^2\} + \frac{2\gamma^2}{u^2}\left[1 + \frac{3}{2}\left(\frac{a\mu}{u}\right)^2 + \cdots\right]\right\}. \quad (57)$$

We see that if $-u/\gamma$ is large compared to 1 but still small compared to γ/a, then the first term of equation (57) dominates. It is just the asymptotic form of equation (55). For $-u/\gamma$ large compared to γ/a, however, the first term becomes exponentially small (due to the fact that μ^2 is not pure imaginary) and the second term (which is itself tending towards zero) dominates. The first term in the brace of equation (57) is proportional, in magnitude to

$$\sqrt{\pi}2x \exp(-a^2x^2/4\gamma^2)$$

which has a maximum at

$$x = \sqrt{2}\gamma/a,$$

the maximum amplitude being

$$2(\gamma/a)\sqrt{2\pi/e}.$$

We can now sketch the form of the real and imaginary parts of $1/\mu^2 Z'(u/\mu a)$ as functions of u (Fig. 1) and hence the form of the Nyquist plot, i.e. the map of the real u-axis in the plane of $\mu^{-2}Z'(u/\mu a)$ (Fig. 2). As we go from $u = +\infty$ towards $u = -\infty$ along the real u-axis (the *opposite* direction from that indicated by the arrows in Fig. 2), the image point starts from the origin, moves outward in a gradually-widening, clockwise spiral until it reaches a radius of order $(a/\gamma) \times (2\pi/e)^{1/2}$, then quickly spirals back into the origin. The dispersion equation (52) will have roots in the upper half-plane (leading to growing waves) if this spiral includes (at least once) the point k^2a^2/ω_p^2. This will happen if

$$\gamma \geq a(k_D/k)^2, \qquad k_D \equiv \omega_p/a. \quad (58)$$

In order for the large field approximation to be valid, we must simultaneously have $\gamma \gg a$. This, combined with equation (58), gives as a condition for instability

$$(k/k_D)^2 nma^2 \ll E_e^2 \ll nma^2(k_D/k)^2, \quad (59)$$

a condition which can always be satisfied, for non-vanishing E_e, at a sufficiently large wavelength.

This is not a physically reasonable result, since it predicts that an electron plasma with a Maxwellian distribution will have some exponentially growing waves no matter how small the applied electric field. When the ions are assumed to have a finite mass, it is not surprising that the same disease manifests itself and one finds growing waves for an arbitrarily small relative drift velocity. The reason for this difficulty

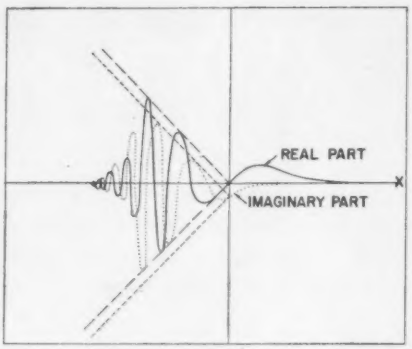

FIG. 1.—Real and imaginary parts of $\mu^{-2}Z'(u/a\mu)$ vs. $x = u/2\gamma$.

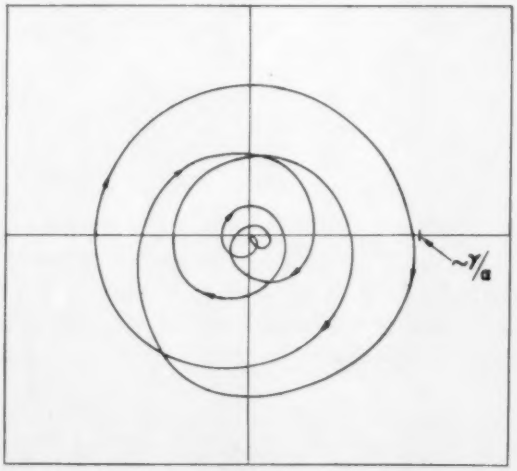

FIG. 2.—Schematic Nyquist diagram for the dispersion equation, equation (52). (The arrows correspond to traversal of the u-axis from $-\infty$ to $+\infty$.)

* A discussion of the asymptotic form of Z is given in the Appendix.

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may be that the original hypothesis is inconsistent; there is no solution for which the random phase approximation and the approximation of nearly constant f_0 are both valid. At any rate, if a solution of the indicated character does exist, then the present results show that f_0 and F_0 must have forms very different from a Maxwellian distribution.

As a final point, we recall the remark, made at the beginning of this section, that the present analysis should describe the completely linear problem of Section 2, at least during a time in which f_0 does not change appreciably. The results found here—instability for any external field—will agree with those of Section 2 only if we can show that the growth *rate* is small compared to eE_e/ma, the rate at which f_0 is changing. Such a demonstration can, in fact, be given so that the results of the two sections are not inconsistent.

CONCLUSIONS

On the basis of the linear analysis of Section 2, we conclude that at least for the special distributions treated there, and probably for more general ones as well, the presence of an external electric field causes no significant changes in the stability character of the linearized plasma waves. If the field is very strong, then it may produce a separation of electron and ion mean velocities greater than the electron thermal speed (thus satisfying the field-free condition for growing waves) before it has carried a particle through one wavelength of the oscillations. In that case, growth is delayed until the particles have gone a distance of order $1/k$ and thus had chance to exchange energy with the plasma waves.

From the results of Section 3 it appears that if a solution of the complete equations in which f_0 and F_0 are nearly constant in time exists, it must involve either an f_0 and F_0 with non-Maxwellian shapes or else must be affected in an important way by the nonlinear terms in equation (4) which couple one mode to another.

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APPENDIX

The plasma dispersion function is defined by

$$Z(z) = \int_{-\infty}^{\infty} \frac{d\theta \exp(-\theta^2)}{\sqrt{\pi}(\theta - z)} \qquad \text{(A1)}$$

for $\mathrm{Im}(z) > 0$. For $\mathrm{Im}(z) < 0$, $Z(z)$ is defined to be the analytic continuation of the function from the upper half-plane of z to the lower half-plane. By expressing $(\theta - z)^{-1}$ as a Fourier integral,* equation (A1) can be transformed to

$$Z(z) = 2\mathrm{i} \exp(-z^2) \int_{-\infty}^{\mathrm{i}z} \exp(-t^2)\, dt \qquad \text{(A2)}$$

a representation valid for both signs of $\mathrm{Im}(z)$.

For real z, it is convenient to separate the real and imaginary parts of equation (A2), writing it as

$$Z(x) = \mathrm{i}\sqrt{\pi} \exp(-x^2) - 2x\,Y(x) \qquad \text{(A3)}$$

where

$$Y(x) = x^{-1} \exp(-x^2) \int_0^x \exp(t^2)\, dt \qquad \text{(A4)}$$

is real. JACKSON's small and large argument expressions for Y, which we reproduce here,

$$Y(x) = \begin{cases} \sum_{n=0}^{\infty} (2x)^{2n}(-1)^n\, n!/(2n+1)! & x \ll 1 \\ \frac{1}{2x^2} \sum_{n=0}^{\infty} x^{-2n} \left(n - \frac{1}{2}\right)!/\sqrt{\pi} & x \gg 1 \end{cases} \qquad \text{(A5)}$$

then give corresponding expressions for $Z(x)$.

However, a little care must be exercized in obtaining asymptotic expressions for $Z(z)$ when z is not real. We require the following result, due to Stokes (WHITTAKER and WATSON, 1952): *Lemma.* The asymptotic expansion of the error function has the form

$$\int_0^z \exp(-t^2)\, dt \simeq \pm\sqrt{\pi}/2 - \frac{\exp(-z^2)}{2z} \left(1 - \frac{1}{2z^2} + \frac{1 \cdot 3}{2^2 z^4} - \frac{1 \cdot 3 \cdot 5}{2^3 z^6} + \ldots\right) \qquad \text{(A6)}$$

where the upper sign is to be taken if $-\pi/2 < \arg z < \pi/2$, the lower sign if $\pi/2 < \arg z < 3\pi/2$.

Proof: Integration by parts shows that the two integrals

$$I_n^{\pm}(z) = \int_z^{\pm\infty} \frac{\exp(z^2 - t^2)}{t^{2n}}\, dt \qquad \text{(A7)}$$

satisfy the recursion relation

$$I_n^{\pm} = \tfrac{1}{2} z^{-(2n+1)} - (n + \tfrac{1}{2}) I_{n+1}^{\pm} \qquad \text{(A8)}$$

From equation (A8) it follows that

$$I_0^{\pm} = \frac{1}{2}\left[\frac{1}{z} - \frac{1}{2}\frac{1}{z^3} + \frac{3}{2}\frac{1}{2}\frac{1}{z^5} + \ldots + A_n/z^{2n+1} + 2(-1)^{n+1} \frac{(n + \tfrac{1}{2})!}{\sqrt{\pi}} I_{n+1}^{\pm}\right] \qquad \text{(A9)}$$

where

$$A_n = (-1)^n\, (n - \tfrac{1}{2})!/\sqrt{\pi}.$$

* This was suggested by E. S. WEIBEL.

To establish the validity of equation (A9) as an asymptotic expansion, we must show (WHITTAKER and WATSON, 1952) that

$$z^{2n+1} I_{n+1} \to 0 \text{ as } |z| \to \infty \tag{A10}$$

We choose a path of integration for $I_n^{\pm}$ in equation (A7) which starts at $t = z = x + iy$ and runs parallel to the real axis to $\pm\infty + iy$. (The contribution from the vertical strip of length y at $\pm\infty$ vanishes.) With $t = s + iy$ we have

$$I_n^{\pm}(z) = \int_x^{\pm\infty} \frac{\exp\{z^2 - (s+iy)^2\}}{t^{2n}}\, ds$$

$$= \int_x^{\pm\infty} \frac{\exp\{x^2 - s^2 + 2iy(x-s)\}}{t^{2n}}\, ds.$$

If $x > 0$, then $x < s$ in the integrand for I_n^+ and so

$$|I_n^+| < \int_x^{\infty} \frac{ds}{t^{2n}} = \frac{1}{2n-1}\left|\frac{1}{z^{2n-1}}\right| \tag{A11}$$

From equation (A8) we have

$$|I_n^+| \leq \tfrac{1}{2}\frac{1}{|z^{2n+1}|} + (n+\tfrac{1}{2})|I_{n+1}^+|$$

which with equation (A11) gives

$$|I_n^+| < \frac{1}{|z^{2n+1}|} \text{ for } x > 0. \tag{A12}$$

Therefore,

$$\left.\begin{aligned} &z^{2n+1} I_{n+1}^+ \to 0 \quad \text{for} \quad |z| \to \infty, \quad x > 0. \\ &\text{Similarly,} \\ &z^{2n+1} I_{n+1}^- \to 0 \quad \text{for} \quad |z| \to \infty, \quad x < 0. \end{aligned}\right\} \tag{A13}$$

To establish the lemma, we note that from equations (A9) and (A13),

$$\left.\begin{aligned} &I_0^+ = \int_z^{\infty} \exp(z^2 - t^2)\, dt \simeq \frac{1}{2}\left[\sum_0^{\infty} A_n / z^{2n+1}\right] \\ &\text{for } x > 0, \text{ where } \simeq \text{ denotes an asymptotic expansion,} \\ &\text{while for } x < 0 \text{ we write} \\ &I_0^+ = \int_{-\infty}^{\infty} \exp(z^2 - t^2)\, dt + \int_z^{-\infty} \exp(z^2 - t^2)\, dt \\ &= \sqrt{\pi}\exp(z^2) + I_0^- \simeq \sqrt{\pi}\exp(z) + \frac{1}{2}\sum_0^{\infty} A_n / z^{2n+1} \end{aligned}\right\} \tag{A14}$$

Thus

$$\int_z^{\infty} \exp(z^2 - t^2)\, dt \simeq [\sqrt{\pi}\exp(z^2)] + \frac{1}{2}\sum_0^{\infty} A_n / z^{2n+1} \tag{A15}$$

where

$$A_n = (-1)^n (n - \tfrac{1}{2})!/(-\tfrac{1}{2})!$$

and the bracketed term is included if $x < 0$, omitted if $x > 0$. This completes the proof, since (A6) follows from (A15) by using

$$\int_0^z \exp(z^2 - t^2)\, dt = (\sqrt{\pi}/2)\exp(z^2) - \int_z^{\infty} \exp(z^2 - t^2)\, dt.$$

However, the form equation (A17) is actually more convenient for our purposes.

Thus, with $z = i\xi$ we have from equations (A2) and (A14)

$$Z(z) = 2i\exp(\xi^2)\int_{-\infty}^{-\xi} \exp(-t^2)\, dt$$

$$= 2i\int_{\xi}^{\infty} \exp(\xi^2 - t^2)\, dt$$

$$Z(z) \simeq [2i\sqrt{\pi}\exp(-z^2)] - \Sigma A_n (-1)^n / z^{2n+1} \tag{A16}$$

where the bracketed term is included if $\mathrm{Re}(z/i) = \mathrm{Im}(z) < 0$, and is omitted if $\mathrm{Re}(z/i) = \mathrm{Im}(z) > 0$. For $\mathrm{Im}(z) = 0$ it would appear that both forms of (A13) are satisfied and that it is equally correct to include the bracketed term in (A15) or to omit it. This is in fact the case, for with z real and $z \to \infty$, the error committed in stopping at any term $1/z^{2n+1}$ is greater than $\exp(-z^2)$. Nevertheless, it is important for dispersion applications to know the correct form of the imaginary part of $Z(x)$, even if it is negligible compared to the real part. We see from equation (A3) that for z real,

$$\mathrm{Im}(Z(x)) = i\sqrt{\pi}\exp(-x^2)$$

while comparison of equations (A3) and (A16) shows that

$$\mathrm{Re}(Z) = -2x\,Y(x) \simeq -\Sigma A_n (-1)^n / x^{2n+1}$$

giving the asymptotic expansion for $Y(x)$ quoted in equation (A5).

To summarize, for $|z| \to \infty$, Z has the asymptotic expansion

$$Z(z) \simeq \sigma i\sqrt{\pi}\exp(-z^2) - \sum_{n=0}^{\infty} \frac{(n-\frac{1}{2})!}{\sqrt{\pi}} \frac{1}{z^{2n+1}} \tag{A17}$$

where $\sigma = 2$ if $\mathrm{Im}(z) < 0$, $\sigma = 1$ if $\mathrm{Im}(z) = 0$, and $\sigma = 0$ if $\mathrm{Im}(z) > 0$.

Of course, it should not be inferred from (A17) that $Z(z)$ has a discontinuity at the real axis, since it is by definition an analytic function. The significance of the equation is rather as follows. For $-y \geq x \gg 1$, the term $\sigma i\sqrt{\pi}\exp(-z^2)$ may be quite large and must be included. For $y \geq x \gg 1$ no such exponential term should be included. For $|y| \ll x$ things are more complicated. Right on the axis ($y = 0$) the entire imaginary part of Z comes from the exponential. In a sufficiently close neighbourhood of the axis, the exponential will still provide a larger contribution to Im (Z) than will the first N terms of the series, but the size of this neighbourhood shrinks as x increases.

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AN EXTENDED ADIABATIC INVARIANT

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Abstract—The motion of a one dimensional oscillator with time dependent restoring force is investigated. We consider $\omega(t)$ having one singularity for finite t and such that

$$\lim_{|t|\to\infty}\left|\frac{d}{dt}\omega^{-1}(t)\right| = 0$$

and consider the asymptotic behaviour of ω times the square of the amplitude. The ratio of this quantity evaluated at two times, t_1 and t_2 averaged over an ensemble is, in the limit $|t_1|, |t_2| \to \infty$, an adiabatic invariant in the sense that this quantity is not changed by multiplication of ω by a slowly varying function of t. This permits replacement of the actual $\omega(t)$ by a, presumably simpler, $\omega_0(t)$ if their ratio, ω_0/ω is non-singular.

Since the orbit equation for a charge in a time-dependent axisymmetric magnetic field is reducible to that of a one-dimensional oscillator it possesses such an invariant. This problem is examined in some detail, particularly for the model field $\omega(t) = \dot{\omega}t$. These solutions provide explicit estimates for the energy gained by a charge under a reversal of the magnetic field.

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1. INTRODUCTION

IN the analysis of charged particle motion in time-dependent fields the notion of adiabatic invariance plays a prominent and useful role. When the fields vary slowly in time these invariants provide a concise specification of all the physically important properties of the orbits. Recently, considerable effort has been put into clarification of the nature of these invariants by BERKOWITZ and GARDNER (1957), HERTWERK and SCHLUTER (1957), KRUSKAL (1957), KULSRUD (1957) and LENARD (1959). Roughly speaking, it is found that for an almost-periodic system whose Hamiltonian changes slowly in time, the action integral is an approximate constant of the motion. If one seeks to exhibit the change in action as expansion in powers of a 'slowness parameter' which measures the fractional change in the external forces during a period, it is found that these corrections vanish to all orders.*

The problem is much more complicated when the Hamiltonian varies rapidly. There exist no simplifying features which connect the instantaneous value of (say) the particle energy with the external parameters, but the motion depends in a complicated way upon the initial conditions and the detailed time history of the forces. The subject of this paper is the study of a simple system under the action of forces which vary rapidly for a finite time, but whose motion becomes asymptotically periodic. First a class of problems of this type is exhibited for which closed expressions for the orbit can be obtained. A study of the asymptotic forms of the orbits reveals a property of the motion which is unchanged by addition of a slowly varying perturbation and leads to an extended adiabatic invariant.

Consider the motion of a one-dimensional harmonic oscillator with time-dependent restoring force. The motion is given by

$$\ddot{x} + \omega^2(t)\, x = 0. \tag{1}$$

If the change in ω during a cycle is small; i.e. if

$$\left|\frac{d}{dt}\omega^{-1}\right| \ll 1 \tag{2}$$

the solution of (1) is approximated by

$$x(t) = Z(t)\sin\left(\int^t \omega(t')\,dt'\right) \tag{3}$$

where $Z^2(t)\omega(t)$ is slowly varying. It has been shown by KULSRUD (1957) that the quantity $Z^2(t)\omega(t)$ is invariant to as many orders in the 'slowness parameter' as $\omega^{-1}(t)$ has continuous derivatives, so that this product defines an adiabatic invariant.

What is sought here is some extension of the idea of adiabatic invariance which is applicable to singular $\omega(t)$. (For the purposes of this discussion $\omega(t)$ will be considered singular wherever $\omega^{-1}(t)$ is not continuously differentiable.) In the next section solutions of

* The paper of VAN VLECK (1926) contains an extensive discussion of the early work on adiabatic invariance, including a simple proof of 'invariance to all orders.' See particularly Section 63.

(1) are exhibited for a class of singular $\omega(t)$. It is found that if $\omega(t)$ is slowly varying at times t_1 and t_2, but not necessarily in the interval, there exists a simple relation between the amplitudes of the motion at these two times. The ratio of these amplitudes turns out to be a simple functional of $\omega(t)$. The result suggests a restatement of the adiabatic invariance concept which may be roughly formulated in the following way. (A more precise formulation will be given in Section 3.) Consider the motion of an oscillator with frequency $\omega(t)$ such that at times t_1 and t_2 ω satisfies $\dot{\omega}/\omega^2 \ll 1$. Suppose another function $\omega_0(t)$ is known, which also has this property, and for which the particle motion is known. We inquire to what extent our knowledge of this model system is useful in describing the actual motion. What is found is that if for the model orbit the ratio of the amplitudes at t_1 and t_2 is known, the corresponding ratio for the actual orbit is related to it only through the instantaneous values of ω and ω_0 at t_1 and t_2 provided that ω_0/ω is slowly varying throughout the interval. If the model field, ω_0, is constant we recover the well-known adiabatic theorem, that for slowly varying ω the amplitude is a function of ω alone. By reformulating the question in this way, however, we are enabled to study situations in which $\dot{\omega}/\omega^2$ is not always small. For example, suppose ω is a reasonably smooth function of time which passes through zero during the motion, so that $\dot{\omega}/\omega^2$ is unbounded. Suppose, however, that we can find an $\omega_0(t)$ which has a zero of the same order as ω, and such that ω_0/ω is slowly varying. Then the knowledge of the orbit under $\omega_0(t)$ enables us to infer a relation between the initial and final amplitudes of the actual orbit even though near $\omega = 0$ the adiabatic approximation fails completely. The essentially new feature of this viewpoint is that the quantity under consideration is not a *local* property of the motion (amplitude, energy, etc.) but an *integral* property, the ratio of amplitudes at two times, which is a function of the entire history of $\omega(t)$ in the interval.

2. EXACT SOLUTIONS

Consider equation (1) with $\omega(t)$ of the form*

$$\omega(t) = kt^\alpha, \quad \alpha > -1$$
$$\omega(-t) = \pm\omega(t). \tag{4}$$

This ω is singular at $t = 0$, but

$$\lim_{|t|\to\infty} \frac{d}{dt}\,\omega^{-1}(t) = 0. \tag{5}$$

The solutions are expressible in terms of Bessel functions of fractional order. A fundamental system of solutions, normalized to unit Wronskian is

$$u(t) = (\pi\nu t)^{1/2} J_\nu(\nu k t^{\alpha+1})$$
$$v(t) = (\pi\nu t)^{1/2} Y_\nu(\nu k t^{\alpha+1}) \tag{6}$$
$$\nu = 1/(2\alpha + 2).$$

The singularity in $\omega(t)$ is reflected in the branch point of $v(t)$ at the origin. In connecting the solutions for positive and negative t, care must be taken to use the correct analytic continuation (WATSON, 1952), and it is in fact here that all the non-adiabatic effects manifest themselves.

The general solution of equations (1) and (4) is, of course, expressible as $x(t) = au(t) + bv(t)$ with constants a and b to be determined from the initial conditions imposed at t_0. To study the solutions for large $|t|$ we replace the Bessel functions by their asymptotic forms finding that u and v reduce to the form (3) and

$$\lim_{|t|\to\infty} |\omega(t)|\, Z^2(t) = (a + 2b\eta \cot \pi\nu)^2 + b^2 \tag{7}$$

where

$$\eta = 0 \quad \text{if } t/t_0 > 0$$
$$\eta = 1 \quad \text{if } t/t_0 < 0.$$

The term $2b\eta \cot \pi\nu$ arises from the continuation of v through the origin.†

The constants a and b are easily obtained from the initial conditions, and will in general be strongly dependent upon the value of t_0. Let us consider, however, an ensemble of systems all having the same initial x and $\dot{x}$ but started at various times close to t_0 and then average over a period $\omega^{-1}(t_0)$. In the limit $|t_0| \to \infty$ we find $\langle a^2\rangle = \langle b^2\rangle = Z^2(t_0)|\omega(t_0)|$ and $\langle ab\rangle = 0$, where $\langle\ \rangle$ indicates the ensemble average. Equation (8) then gives the simple result

$$K\{\omega\} \equiv \lim_{|t|,\,|t_0|\to\infty} \left\langle \frac{|\omega(t)|\, Z^2(t)}{|\omega(t_0)|\, Z^2(t_0)} \right\rangle = 1 + 2\eta \cot^2 \pi\nu. \tag{8}$$

The first relation defines K which is a functional of $\omega(t)$.

The interesting property of $K\{\omega\}$ is that it depends upon $\omega(t)$ only through the parameter ν and not upon the coefficient, k. ν describes the nature of the singularity at $t = 0$. In the limit $\alpha \to 0$ we find, of course, $K\{\omega\} = 1$. The case of greatest practical interest is that with $\alpha = 1$ where $\nu = 1/4$ and $K = 3$.

* The case $\alpha = 1/2$ is familiar from the WBK connexion formulas in quantum mechanics.

† Similar continuation formulae for a somewhat more general class of problems have been given by GOLDSTEIN (1928).

We note in passing that while the sign of ω does not appear in the above discussion it does not follow that for the related problem of a charge in a magnetic field the sign of the field is of no consequence. Indeed, the sign of ω plays a crucial role as will be seen in Section 4.

3. AN EXTENDED ADIABATIC INVARIANT

Consider the class of $\omega(t)$ satisfying condition (5). Let a member of this class, $\omega_0(t)$, have at most one singularity, which we, for convenience, choose to be the origin of time. Let another $\omega(t)$ be well approximated by ω_0 in the sense that $\omega_0(t)/\omega(t)$ is regular at $t = 0$ and slowly varying everywhere, and in the limit $|t| \to \infty$ all derivatives of $\omega_0(t)/\omega(t)$ vanish.* Assuming $K\{\omega_0\}$ to be known, we seek to express $K\{\omega\}$ as $K\{\omega_0\}$ plus a power series in a slowness parameter, ε, which measures the rate of change of ω_0/ω.

Let $x(t)$ be written in the form

$$x(t) = W(t)x_0[y(t)] \tag{9}$$

where W, x_0, y are to be defined in due course. Equation (1) becomes

$$(\ddot{W} - \omega^2 W)\, x_0(y) + (2\dot{W}\dot{y} + W\ddot{y})x_0'(y) + W\dot{y}^2 x_0''(y) = 0 \tag{10}$$

where the primes indicate differentiation with respect to y. We now introduce a function $q(y)$ which is *defined* by

$$q\left(\int_0^t \omega(t')\, dt'\right) = \omega_0(t),$$

and then choose $x_0(y)$ to be a solution of

$$x_0''(y) + \frac{q'(y)}{q(y)}\, x_0'(y) + x_0(y) = 0. \tag{11}$$

The definitions of q and x_0 are so constructed that if $\omega = \omega_0$ equation (10) is trivially satisfied by $y = \int^t \omega_0(t')\, dt'$ and $W =$ constant. Our problem is now to find functions of $y(t)$ and $W(t)$ which satisfy equation (10) for general $\omega(t)$.

The following development is a straightforward extension of the technique of KULSRUD and will be sketched rather briefly. Eliminating x_0'' from equations (10) and (11) we equate separately to zero the coefficients of x_0 and x_0' to obtain

$$\ddot{W} + \omega^2 W - \dot{y}^2 W = 0 \tag{12}$$

$$2\dot{W}\dot{y} + W\ddot{y} - \dot{y}^2 W q'/q = 0. \tag{13}$$

Equation (13) may be integrated at once to give

$$\frac{W^2\dot{y}}{q} = \text{constant}. \tag{14}$$

Assuming ω_0/ω to be slowly varying, let ε represent a characteristic rate of change so that ω_0/ω is a function of $\tau = \varepsilon t$. Equation (12) may be rewritten

$$\varepsilon^2 \frac{d^2 W}{d\tau^2} + (\omega^2 - \dot{y}^2)W = 0. \tag{15}$$

We solve equations (14) and (15) by the substitution

$$W(\tau) = \sum_n \varepsilon^n W_n(\tau); \quad y(t) = \sum_n \varepsilon^n y_n(t) \tag{16}$$

with $y_0 = \int_0^t \omega(t')\, dt', \quad y_n(0) = 0, \quad W_0^2(\tau) = \omega_0/\omega.$

It is also convenient to expand $q(y)$ in a Taylor series about $y = y_0$;

$$q(y) = \sum_{m=0}^{\infty} \frac{q^{(m)}(y_0)(y - y_0)^m}{m!}. \tag{17}$$

We postpone for later discussion any questions concerning the convergence of this expansion.

The procedure is now to substitute expansions (16) and (17) into equations (14) and (15) and equate separately to zero the coefficients of successive powers of ε. While the general expression for the coefficients is quite complicated and will not be written down, the pattern of solution is clear. Examining the coefficient of ε^n in equation (15) we find that, because of the choice of W_0, y_0, the highest order W_r which appears is $r = n - 2$. Hence, if all W_r, y_r are known for $r < n$, equation (15) provides an explicit expression for y_n. Substituting into equation (14) gives W_n, and by this method the complete expansion is obtained. Observe that in each step we obtain y_n and W_n in terms of the first n derivatives of ω_0/ω. Since as $|t| \to \infty$ all these derivatives tend to zero by hypothesis we have asymptotically $W = W_0$; $y = y_0 +$ constant. It follows, therefore, that if $x(0)$ and $\dot{x}(0)$ are specified, the quantity $\lim_{|t|\to\infty} W^2(t) = \lim_{|t|\to\infty} |\omega(t)|Z^2(t)$ is the same for both ω and ω_0 to as many orders in ε as ω_0/ω has continuous derivatives.

Thus far, we have proved that, given x and $\dot{x}$ at $t = 0$, $\lim_{|t|\to\infty} \omega(t)Z^2(t)$ is invariant under smooth changes of $\omega(t)$. However, if the initial conditions are to be specified at $t_0 \neq 0$ the quantity $y(t_0)$ is an essentially unknown function of $\omega(t)$. Hence, while a fundamental system of solutions for both $\omega(t)$ and $\omega_0(t)$ is known, the particular linear combination appropriate to the initial conditions is not. This difficulty is circumvented by averaging over initial

* Clearly we may equally well compare the orbits at two finite times t_1 and t_2 if at both times all derivatives of ω_0/ω vanish. We consider the case $|t_1|, |t_2| \to \infty$ only to retain the notation of the preceding section.

phases. Given an orbit for $\omega_0(t)$ belonging to initial conditions $x(t_0)$, $\dot{x}(t_0)$ we may find another time t_0' such that $|\omega(t)|Z^2(t)/|\omega(t_0')Z^2(t_0') = |\omega_0|(t)Z_0^2(t)/|\omega_0|(t_0)Z_0^2(t_0)$ in the limit as $|t| \to \infty$. Since for large $|t_0|$, t_0 and t_0' differ by a constant phase, we average over initial phases to obtain $K\{\omega\} = K\{\omega_0\}$.

This completes the proof except for the question of convergence of expansion (17). From the definition of q we see that if ω_0 is singular at $t = 0$ there is a corresponding singularity in $q(y)$. For the simple case of a sign change in ω, q is a double-valued function of y and hence has a branch point at the origin. Hence, to insure convergence, we must verify that $|y(t) - y_0(t)| < |y(t)|$, since the origin is assumed to be the only singularity of ω_0. For each sign of t, $\omega_0(t)$ has a definite sign so that q is a definite function of y. By hypothesis ω_0/ω is also smooth, so that ω must also be definite on each half axis, and from equation (14) it follows that y has the same sign as y_0. Hence, the expansion converges whenever $|y| < 2|y_0|$. When this inequality fails (and there seems to be no reason for it to hold) we must modify the expansion. From equation (14) we see that y/y_0 is bounded, so that there exists a constant $M > 0$; such that the Taylor expansion of q in powers of $(y\text{-}My_0)$ converges. It is easy to verify that if equation (17) is modified in this way no change is required in the remainder of the argument.

The theorem on extended adiabatic invariance may be stated as follows: Let $\omega(t)$ and $\omega_0(t)$ satisfy equation (5), $\omega_0(t)$ have at most one singularity on the real axis and

$$\lim_{|t|\to\infty} \frac{d^n}{dt^n}\frac{\omega_0}{\omega} = 0; \text{ then } K\{\omega\} - K\{\omega_o\} = 0$$

to as many orders in ε as ω_0/ω has continuous derivatives. In this sense the quantity $K\{\omega\}$ is an extended adiabatic invariant.

4. APPLICATION TO CHARGED PARTICLE MOTION IN AN AXISYMMETRIC MAGNETIC FIELD

The previous discussion related only to the one-dimensional harmonic oscillator with time dependent restoring force. We now consider a related two dimensional problem in which these results are applicable. It is well known that the motion of a charge in a slowly varying field is such that its magnetic moment is an adiabatic invariant, or alternatively, that its energy is proportional to the field strength. However, if the field changes rapidly and particularly if it changes sign the particle is subjected to violent acceleration and large energy changes are possible. With the aid of the extended invariant it is possible to obtain a simple connexion between the initial and final energies of an ensemble of charges experiencing a sudden change in magnetic field.

Consider an axisymmetric system with magnetic field, B, in the z direction and independent of r out to the largest radius explored by the orbit under consideration. The equations of motion are

$$p_\theta = mr^2\left(\dot{\theta} + \frac{\omega}{2}\right) = \text{constant} \tag{18}$$

and

$$\ddot{r} + \frac{\omega^2}{4}r = \left(\frac{p_\theta}{m}\right)^2 \Big/ r^3 \tag{19}$$

where $\omega(t) = eB(t)/mc$. Equation (18) expresses the conservation of the generalized angular momentum, p_θ, while (19) gives the radial motion. In the subsequent discussion it will be convenient to describe the orbit in terms of an instantaneous guiding centre co-ordinate, R, and cyclotron radius, λ. These are most conveniently defined in terms of the vector relations $\bar{\lambda} = \dfrac{\bar{v} \times \bar{\omega}}{\omega^2}$, $\bar{r} = \bar{R} + \bar{\lambda}$, where $\bar{\omega}$ is in the direction of B. In these variables, equation (18) may be written

$$\frac{p_\theta}{m} = \frac{\omega}{2}(R^2 - \lambda^2) = \text{constant.} \tag{18a}$$

We remark here that equation (18a) is a rigorous consequence of the axial symmetry of the system and does not depend for its validity upon any approximations used in solving equation (19).*

Solution of equation (19) is facilitated by a transformation which eliminates the non-linear term. Let

$$\mu(t) = r(t)\cos\left\{\frac{p_\theta}{m}\int_{t_0}^{t}\frac{dt'}{r^2(t')}\right\} \tag{20}$$

which reduces equation (19) to

$$\ddot{\mu} + \frac{\omega^2(t)}{4}\mu = 0. \tag{21}$$

The inverse to (20) is

$$r(t) = \mu(t)\left\{1 + \left(\frac{p_\theta}{m}\int_{t_0}^{t}\frac{dt'}{\mu^2(t')}\right)^2\right\}^{1/2}. \tag{22}$$

Transformations (20) and (22) reduce the general orbit problem to that of $p_\theta = 0$.

The required integral in equation (22) may be carried out at once in terms of the solutions of (21).

* Note that the equation of motion may be put into covariant form simply by replacing t by the proper time. Then in the asymptotic region the total energy is related to the cyclotron radius by

$$\left(\frac{E}{mc^2}\right)^2 = \gamma^2 = \frac{1}{2}\left\{1 + \sqrt{\left[1 + \left(\frac{2\omega\lambda}{c}\right)^2\right]}\right\}.$$

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Let μ be a solution of (21) satisfying the appropriate initial conditions, and let $\mu^+(t)$ be another solution which is linearly independent of μ, but otherwise arbitrary. The Wronskian $W\{\mu, \mu^+\}$ is time independent and it is easily verified that

$$\int^t \frac{dt'}{\mu^2(t')} = \frac{1}{W\{\mu, \mu^+\}} \frac{\mu^+(t)}{\mu(t)} .$$

A convenient choice of μ^+ is given by $\mu^+(t_0) = 0$, $\dot{\mu}^+(t_0) = 1$. Then $W\{\mu, \mu^+\} = r_0$, and equation (22) becomes after squaring both sides

$$r^2(t) = \mu^2(t) + \left(\frac{p_\theta}{mr_0}\, \mu^+(t)\right)^2. \tag{23}$$

In the asymptotic region, where $\omega(t)$ is slowly varying, μ and μ^+ reduce to trigonometric functions. Writing

$$\mu(t) \approx Z_1(t) \cos\left(\frac{y + \phi_1}{2}\right),$$

$$\mu^+(t) \approx Z_2(t) \cos\left(\frac{y + \phi_2}{2}\right)$$

where, again, $y = \int^t \omega(t')\, dt'$. Equation (23) becomes

$$r^2(t) \to R^2(t) + \lambda^2(t) - 2R\lambda \cos(y + \psi) \tag{24}$$

where

$$R^2 + \lambda^2 = \frac{Z_1^2}{2} + \frac{Z_2^2}{2}\left(\frac{p_\theta}{mr_0}\right)^2 \tag{24a}$$

$$2R\lambda \cos(y + \psi) = \frac{Z_1^2}{2}\cos(y + \phi_1) + \frac{Z_2^2}{2} \times \left(\frac{p_\theta}{mr_0}\right)^2 \cos(y + \phi_2). \tag{24b}$$

Since μ and μ^+ are solutions of equation (21) with specified initial conditions, the results of the previous section apply separately to Z_1 and Z_2. It follows at once that

$$L\{\omega\} \equiv \lim_{|t|,\, |t_0| \to \infty} \left\langle \frac{|\omega(t)|\,[R^2(t) + \lambda^2(t)]}{|\omega(t_0)|\,[R^2(t_0) + \lambda^2(t_0)]} \right\rangle \tag{25}$$

is an extended adiabatic invariant. In the case $\omega = kt^\alpha$ we have $L\{\omega\} = 1 + 2\eta \cot^2 \pi\nu$.

[For fields such that $\omega(t) \to \infty$ as $|t| \to \infty$, if $r(t_0)$, $\dot{r}(t_0)$ are fixed and we examine equation (24a) in the limit $|t_0| \to \infty$, it is found by explicit calculation that the phase dependent terms cancel exactly, so that the ensemble average in equation (25) is not required. This cancellation is not accidental but has a simple physical meaning. If r_0 and $\dot{r}_0$ are fixed, but $\omega(t_0)$ is allowed to increase without bound, the cyclotron radius, λ, tends to zero while the kinetic energy remains finite. This is to be contrasted with the linear harmonic oscillator, where the amplitude of the motion is greater than r_0 and the energy tends to infinity as $\omega(t_0)$ grows without bound. If a more general class of initial conditions is to be considered, i.e. if we allow r and $\dot{r}$ to be functions of t_0 so that the initial amplitude does not vanish in the limit $|t_0| \to \infty$, then the phase average is essential.]

Given an ensemble of charges started with the same R and λ, $L\{\omega\}$ connects the average value of R^2 and λ^2 in the asymptotic regions. Combining with equation (18a) gives $\langle R^2 \rangle$ and $\langle \lambda^2 \rangle$ separately at the final time. Notice that while the sign of ω does not appear in the adiabatic invariant, it does enter through p_θ. To illustrate the importance of the sign consider the case $\omega = kt^\alpha$. We find for $\omega(-t) = \omega(t)$

$$|\omega_f|\,\langle R_f^2 \rangle = |\omega_i|\,[\langle R_i^2 \rangle (1 + \cot^2 \pi\nu) + \langle \lambda_i^2 \rangle \cot^2 \pi\nu] \tag{26a}$$

$$|\omega_f|\,\langle \lambda_f^2 \rangle = |\omega_i|\,[\langle R_i^2 \rangle \cot^2 \pi\nu + \langle \lambda_i^2 \rangle (1 + \cot^2 \pi\nu)]$$

while for $\omega(-t) = -\omega(t)$,

$$|\omega_f|\,\langle R_f^2 \rangle = |\omega_i|\,[\langle R_i^2 \rangle \cot^2 \pi\nu + \langle \lambda_i^2 \rangle \times (1 + \cot^2 \pi\nu)] \tag{26b}$$

$$|\omega_f|\,\langle \lambda_f^2 \rangle = |\omega_i|\,[\langle R_i^2 \rangle (1 + \cot^2 \pi\nu) + \langle \lambda_i^2 \rangle \cot^2 \pi\nu].$$

We see that changing the relative signs of $\omega(-t)$ and $\omega(t)$ has the effect of interchanging the roles of R_f and λ_f. An interesting special case is that of $\alpha = 0$, $\omega(t) = -\omega(-t) =$ constant, so that $R_i = \lambda_f$, $R_f = \lambda_i$. This result is obtainable by elementary methods from the impulse applied during the instantaneous field reversal.

A case of practical interest is that of $\alpha = 1$. For charges injected at rest, $\lambda_i = 0$ and the final (non-relativistic) energy is given by $\langle E_f \rangle = m\omega_f R_i^2$. With readily obtainable fields an ion may be accelerated to tens of kilovolts, and an electron to relativistic energies.* To illustrate the effect of the non-adiabatic acceleration, the case $\omega(t) = \dot{\omega}t$ was investigated in detail. Considering a particle injected at time t_0 with $r = r_0$, $\dot{r}(t_0) = \dot{\theta}(t_0) = 0$, μ and μ^+ were found in terms of solutions (6) by fitting the initial conditions, and then $|\omega|R^2$ and $|\omega|\lambda^2$ are readily determined in the asymptotic region from equations (18a) and (24a). In Fig. 1 is plotted the asymptotic value of $|\omega|\lambda^2$ in units of $\dot{\omega}^{1/2} r_0^2$ as a function of the reduced injection time $\tau = \dot{\omega}^{1/2} t_0$. For $\tau \geqslant 1$ the entire motion is

* Compare with the description of the 'Megatron' by FINKELSTEIN (1958). The only difference between the two calculations is that here the betatron condition is not satisfied.

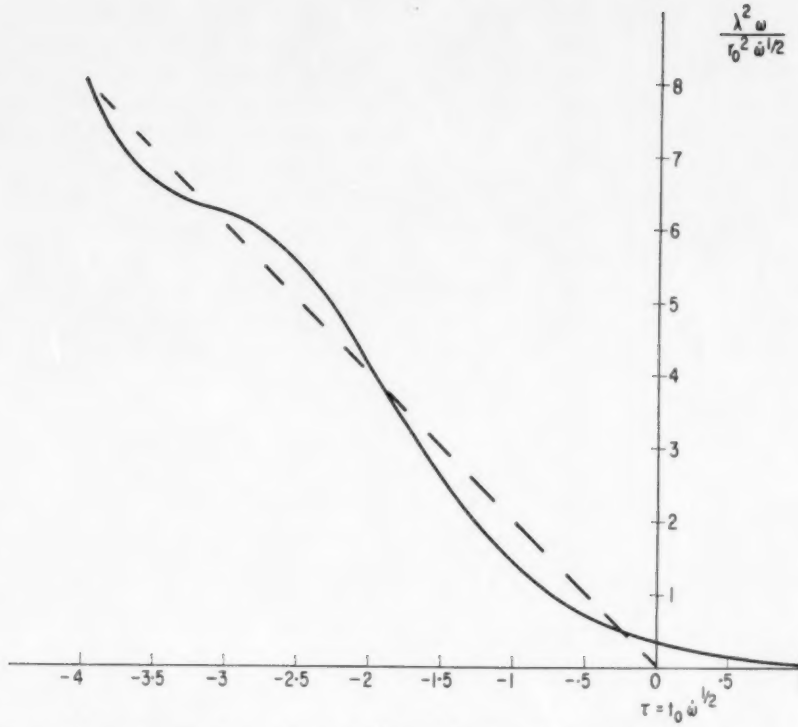

FIG. 1.—Plot of final value of $\omega\lambda^2$ in units of $r_0^2\dot{\omega}^{1/2}$ as a function of the reduced injection time $\tau = t_0\dot{\omega}^{1/2}$ for linear field variation, $\omega = \dot{\omega}t$. Initial conditions are $r = r_0$ and $\dot{r}_0 = \dot{\theta}_0 = 0$. Dashed line indicates asymptotic result from equation (26b).

adiabatic and not much happens. When $\tau \leqslant 1$ the charge is injected during the non-adiabatic interval and the final energy begins to rise. Finally, if $\tau \ll -1$ the asymptotic forms (26) are a good approximation and the final λ^2 increases linearly with $|\tau|$. The physical meaning of this result is simply that, if $\tau \ll -1$ the particle, being tied to a field line, moves radially outward until $\tau \simeq -1$. Then it is subjected to a strong electric field proportional to r and hence proportional to r_0, until $\tau \approx +1$. From then on the particle gains energy by adiabatic acceleration.

The apparent paradox of the asymmetry of results (8), (25), etc. with respect to interchange of initial and final times requires some comment. A consequence of these results is that the final energy of the ensemble is always greater than the initial energy, which is inconsistent with the time reversibility of the orbits. It is easy to verify, however, that this is due to a phase-bunching effect, in which an ensemble uniformly distributed in phase does not remain so. Hence the appropriate initial ensemble for the time reversed problem is not uniform but must depend upon the initial phase.

In any application of the above results it must be kept in mind that the magnetic field has been assumed constant over the volume explored by the test particle. This is a very severe restriction, since it excludes the study of plasmas with large diamagnetic currents. Furthermore, the presence of physical boundaries to the system limits the class of possible orbits. Since those particles which gain the most energy are those which at the time of field reversal are far from the cylinder axis, the walls set an upper limit to the energy gain. To find the maximum energy gain we examine that orbit which just grazes the wall; particles started at too large an initial radius are effectively lost. If the exponent, α, is not too large, the maximum radius is reached very close to $t = 0$ and we find that if the cylinder radius is a, the largest orbit satisfies in the asymptotic region

$$|\omega|\lambda^2 \approx 2(1 + \cot^2 \pi\nu)\frac{\Gamma^2(1-\nu)}{\pi\nu}\left(\frac{\nu k}{2}\right)^2 a^2.$$

If the field varies linearly to some peak value this result is expressible in terms of the voltage around the cylinder and the peak energy per unit length stored in the magnetic field. If L is the energy per unit length in ergs/cm and V the voltage in e.s.u., we find

$$(\omega\lambda)^2 \lesssim \frac{24}{c}\left(\frac{e}{\pi m}\right)^{3/2}\sqrt{LV}.$$

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NEUTRON EMISSION FROM LINEAR PINCHES IN DEUTERIUM AT HIGH RATES OF CURRENT RISE

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Abstract—This paper describes the neutron emission observed during an investigation of the characteristics of fast linear *Z*-pinches using a low-inductance condenser bank. The neutron production is compared with the results obtained elsewhere and is shown to be different in several respects from that obtained with low-power banks, but is quite similar to that reported in more recent, high-power experiments. It is shown that much of the yield is non-thermonuclear, and a number of existing theories for the production of neutrons by acceleration processes in a plasma are discussed. A process of neutron production in fast linear pinches is suggested, which qualitatively accounts for many recently reported results; the proposed mechanism becomes the well-known Colgate process in discharges where $m = 0$ instabilities lead to narrow necks in the plasma. Reasons are given to account for the absence of appreciable thermonuclear yields which might be expected from theoretical considerations.

1. INTRODUCTION

THE possibility of heating a gas to thermonuclear temperatures by means of rapid magnetic compression (ANDRIANOV *et al.*, 1956; BICKERTON, 1957; FITCH, 1959) is being studied at A.W.R.E. Such a rapid compression can be produced most easily in a straight tube using an axial current (*Z*-pinch) and has been studied by many workers (including ARTSIMOVICH *et al.*, 1956; ANDERSON *et al.*, 1958; DUNAWAY and PHILLIPS, 1958; MATHER and WILLIAMS, 1958; BERGLUND *et al.*, 1957; ALLEN *et al.*, 1957; BODIN and REYNOLDS, 1957; CURRAN *et al.*, 1958).

Two features common to all these studies are: the failure to achieve a high enough plasma temperature to give thermonuclear fusion; and the emission of spurious neutrons. The present paper is concerned with the characteristics of the neutron emission from a linear *Z*-pinch with a high rate of rise of current. The results are compared with other experiments and mechanisms of neutron emission are reviewed in an attempt to explain some of the anomalies in the results reported by different workers.

2. APPARATUS

The experiments were carried out with the Maggi condenser bank (FITCH and MCCORMICK, 1959) shown schematically in Fig. 1. The total capacity is 100 μF and the inductance 5 mμH; voltages up to 25 kV were used giving initial rates of current rise in the gas up to 2×10^{12} A/sec.

The discharge tube was of clear quartz (28 cm long, 15 cm bore, 7·5 mm wall thickness) with plane copper electrodes. A continuous-flow gas-filling system was usually used and the deuterium had an impurity content ~0·2 per cent CO_2 and N_2. A low-energy oscillatory discharge (12 kA mean current) was used for pre-ionization to ensure rapid breakdown when the bank was fired. Varying the intensity of the pre-ionizing discharge did not affect the neutron yield.

The voltage, current and rate of change of current were measured in the usual way as shown in Fig. 1 and streak photographs of the discharge were taken. The total neutron emission was monitored by (n, γ) reactions from thermalized neutrons in a 6·3 litre terphenyl-toluene scintillation counter surrounded by cadmium foil, while the time of the neutron emission was measured with a small (10 cm) unshielded counter of the same scintillator, having a response time of 7×10^{-9} sec.

3. PROPERTIES OF THE NEUTRON EMISSION

(1) *General aspects of the neutron production under typical initial conditions of pressure* (50 μ Hg) *and bank voltage* (20 kV)

A typical neutron pulse is shown in Fig. 2. The emission starts at about the first pinch, rises slowly, falls to zero before the current peak, and lasts for 1 to 2 μsec: the total yield is about 10^6 neutrons. The time of onset and duration of the neutron pulse is similar to that observed by HAGERMAN and MATHER (1958) and ANDRIANOV *et al.* (1958) who both used $\dot{I} \sim 10^{12}$ A/sec. In the small bank at A.W.R.E. (OSWALD II, CURRAN *et al.*, 1958) with $\dot{I} \sim 5 \times 10^{11}$ A/sec, the neutron emission began at the third pinch and lasted $\frac{1}{2}$ μsec. All these results are in contrast to those with $\dot{I} \leq 10^{11}$ A/sec (ANDERSON *et al.*, 1958; DUNAWAY

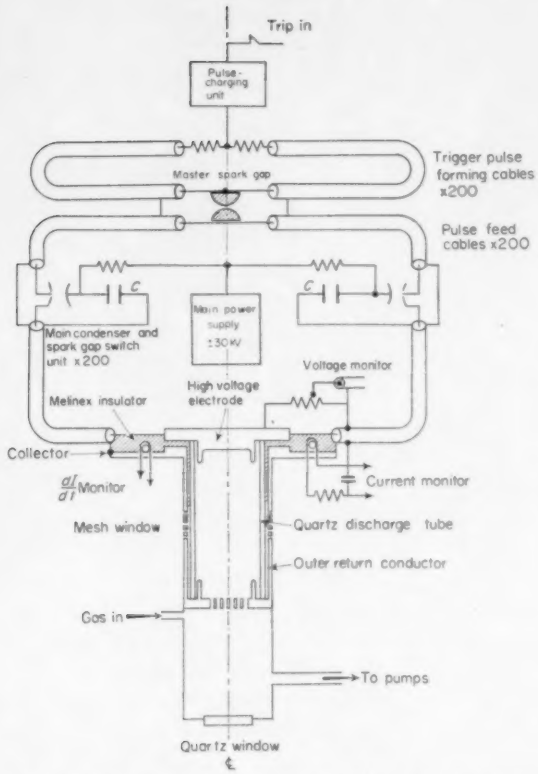

FIG. 1.—Schematic diagram of Maggi *Z*-pinch discharge circuit.

and PHILLIPS, 1958) in which the neutrons ($\sim 10^7$–10^8) were emitted over a very short period ($\sim 10^{-7}$ sec) at the second or third pinch.

The neutron emission is fairly constant from one

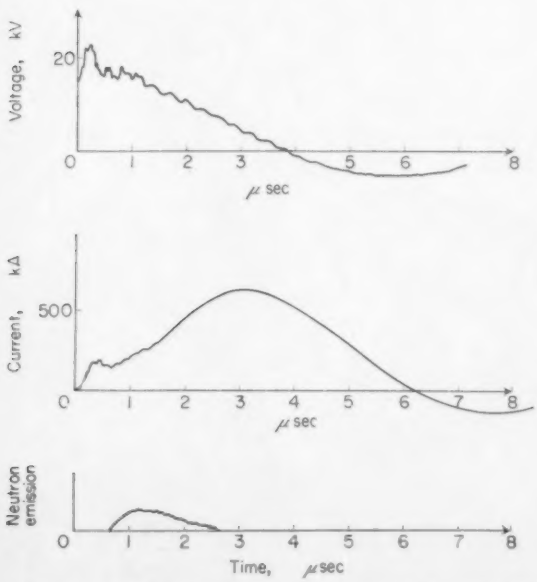

FIG. 2.—Typical *Z*-pinch wave forms showing also the neutron emission.

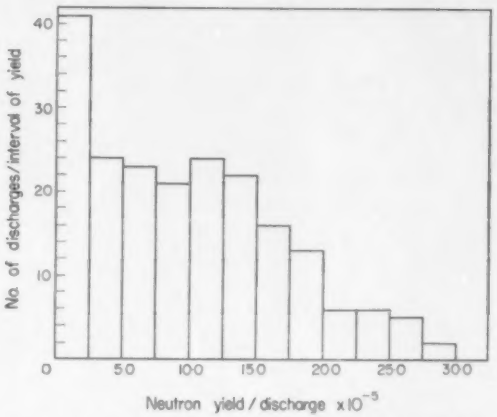

FIG. 3.—Histogram of neutron yield under conditions of constant bank voltage 25 kV, and pressure, 35 μ.

discharge to another as shown in the histogram in Fig. 3; with an average yield of 10^6 neutrons per discharge the yield lay within a factor of 2 of the mean for two-thirds of the discharges. This is in contrast to some early work on typical instability yields (DUNAWAY and PHILLIPS, 1958; ARTSIMOVICH *et al.*, 1956) in which 'large fluctuations' of the yield were reported.

(2) *Variation of the neutron yield with gas pressure and impurity content*

The variation of neutron yield with pressure is shown in Fig. 4. The fall-off in the yield below 30 μ Hg is probably due to inefficient ionization in the initial stages preventing a proper discharge from taking place. Above this pressure the observed dependence is less steep than would be expected from a thermonuclear yield on all simple models of the pinch (e.g. $T \propto I^2/N$, where N is the line density). The neutron yield was not found to be very sensitive to small amounts of impurities, in contrast to ARTSIMOVICH *et al.* (1956), ANDERSON *et al.* (1958) and DUNAWAY and PHILLIPS (1958). In the present experiments the yield was reduced by a factor 2 when the nitrogen impurity was raised from 0·2 per cent to 2 per cent.

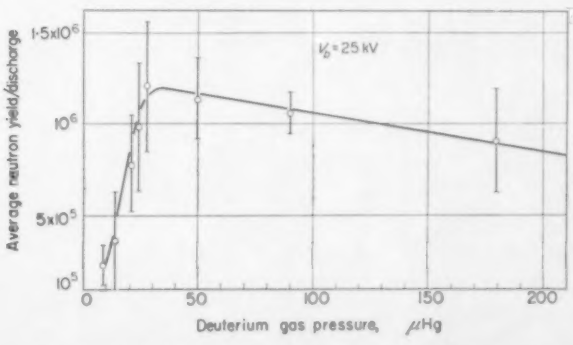

FIG. 4.—Variation of neutron yield with deuterium gas pressure.

(3) *The effect of an axial magnetic field on the neutron production*

In the experiments of ANDERSON *et al.* (1958) axial magnetic fields (H_z) of the order of a 100 gauss were sufficient to prevent the production of neutrons by an $m = 0$ instability process. In order to see if the neutron emission at high rates of current rise occurs when the $m = 0$ mode is stabilized (ROSENBLUTH, 1956) steady magnetic fields up to 2000 gauss were applied.

Fig. 5 shows the yield as a function of H_z; the dotted curve shows the results with the small A.W.R.E. bank (CURRAN *et al.*, 1958). In both cases there is a gradual reduction of the yield as H_z is increased. This is in contrast with the results of MATHER and WILLIAMS (1958) who found that a field of only 60 gauss caused a factor of five reduction in yield, while a

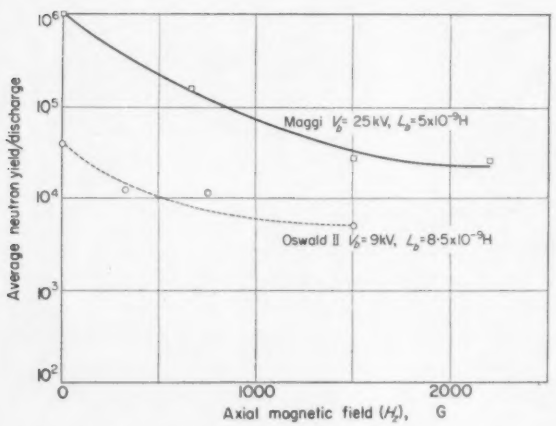

FIG. 5.—Neutron yield vs. mean applied H_z.

further 500 gauss was necessary to reduce the yield by another factor of five. The form of their curve at fields greater than 100 gauss is similar to the whole curve in the present experiments. The present results on the effect of H_z on the total yield are similar to those reported by OHLIN and SIEGBAHN (1958).

The application of fields up to 2000 gauss does not have a significant effect on either the time of emission or the duration of the neutron burst, and the emission always ceases before the current maximum.

(4) *The relationship between the neutron yield and the pinch dynamics: variation of yield with voltage and current*

One can relate the temperature in the dynamic pinch to the implosion velocity, or to the current (by pressure balance) if equilibrium between the containing field and the plasma is achieved. On the former picture the yield should be a very sensitive function of dI/dt and hence bank voltage, while on the latter a

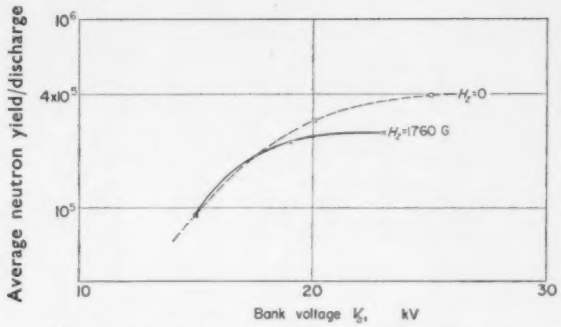

FIG. 6.—Neutron yield versus bank voltage with axial magnetic field H_z as a parameter. $H_z = 0$ normalized to $H_z = 1760$ G at 15 kV.

sharp dependence on the current at the time of emission would be expected. Fig. 6 ($H_z = 0$) shows that the yield is not a very sensitive function of voltage.

The variation of the yield with current was found by varying the capacity, but it should be noted that this also changes the bank inductance. The yield is shown as a function of bank capacity in Fig. 7, ($H_z = 0$) and as a function of the current at the peak of the neutron burst in Fig. 8, where the theoretical thermonuclear yield calculated from the pressure balance temperature is also shown (the assumptions made in this calculation are discussed in Appendix A4). It is seen that the observed yield is not particularly sensitive to bank capacity or current—a roughly linear relationship is found.

The experiments described above were repeated with an axial field of 1750 gauss. If the application of H_z had had the intended effect of suppressing spurious modes of neutron production and enhancing the fraction of the yield due to thermonuclear reactions even although the total yield was less, the yield with H_z should be a much more sensitive function of voltage and capacity. This was not found (see Figs. 6 and 7) except at very low capacitance (see Section 3 (7)).

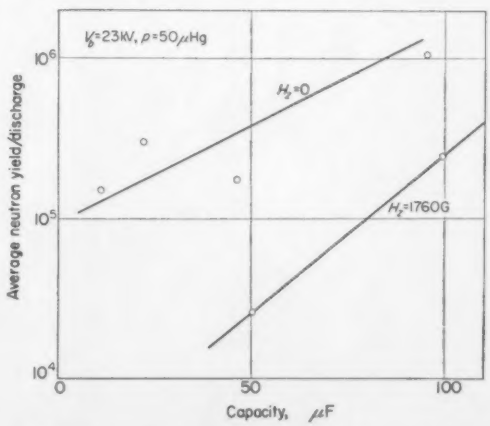

FIG. 7.—Variation of neutron yield with bank capacity.

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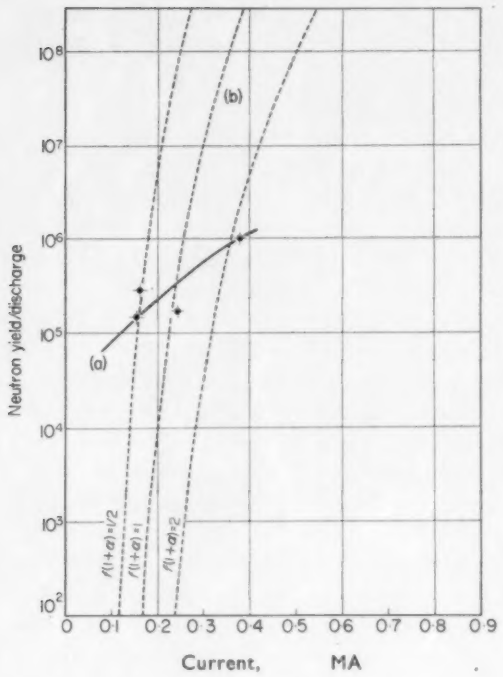

FIG. 8.—Variation of neutron yield with current. (a) Observed yield at the maximum of the prompt neutron pulse $-\overset{\dagger}{\bullet}-$

(b) Theoretical thermonuclear yield obtained from pressure balance considerations, – – – . f = fraction of gas swept up in the pinch.
$\alpha = T_e/T_i$ (see Appendix A (4)).

(5) *Neutron measurements using nuclear emulsions*

Ilford C2 photographic plates were exposed axially a few cm from each end of the tube, and radially midway along its length; the plane of the emulsion was perpendicular to the tube axis for the radial plates. A stabilizing axial magnetic field of 500 gauss was used for the experiment. The average yield per discharge was 7×10^5; some 5000 discharges were fired and 58 proton recoil tracks were counted over an area of 0·2 cm² on the cathode plate and 38 tracks in 0·9 cm² on the anode plate.

Spatial origin of the neutrons. The density of the tracks in the cathode plate was 7·6 times greater than in the anode plate. From this it follows that most of the neutrons were produced near the cathode and the angular distribution measurements (see below) confirmed this. The results are consistent with production of all neutrons about 8 cm from the cathode, or with production of one-third of the total yield at the cathode surface together with production of two-thirds uniformly along the tube.

Angular distribution measurements. If the neutron source subtended a very small angle at the plate, then the energy of the neutrons would be related to the energy of the recoil protons by the expression

$$E_n = \frac{E_p}{\cos^2\theta}$$

where θ is the angle the track makes with the direction of the incident neutron. If the recoil proton energy from a monoenergetic neutron source is plotted as a function of θ, all points lie on the curve $E_p = E_n \cos^2\theta$; departures from this curve will arise due to any variation in the neutron energy. However, for a point source which is not monoenergetic, the curve $E_p = E_n \cos^2\theta$ (where E_n is now the *maximum* neutron energy) should form an envelope which encloses all the points. Fig. 9 shows the angular distribution of the recoil proton tracks; the full curve shows $E_p = E_n \cos^2\theta$ for $E_n = 2{\cdot}6$ MeV (the recoil tracks show that some neutrons of a least 2·6 MeV must have been produced). This curve does not contain nearly all the points, and cannot do so unless an improbably high value of E_n is used. From this it follows that the source is not a point, but must subtend an angle at the plate. The dotted lines show $E_p = 2{\cdot}6 \cos^2\theta$ for sources at angles $\phi = +30°$ and $-30°$ and the three curves together form an envelope enclosing most of the points. This is interpreted as showing a wide source, subtending a solid-angle of 60° at the cathode plate. Fig. 10 illustrates ϕ and θ. A similar plot of recoil energies versus θ for the anode plate gave $\phi = \pm 15°$. Angular distribution measurements in the radial direction also indicated a comparatively wide source.

Energy measurements. Because of the low yields in the present experiments a neutron spectrum could only have been obtained if the source had subtended a small angle at the plates. The longest recoil tracks however set a lower limit to the maximum energy of the neutrons emitted in each direction. On the cathode

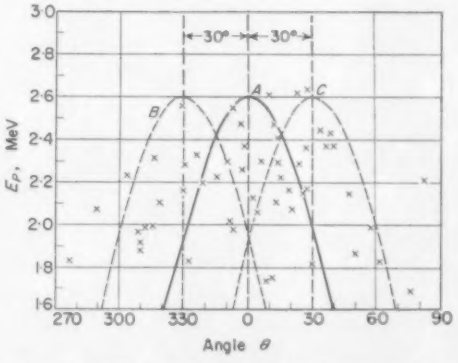

FIG. 9.—Angular distribution of recoil protons on cathode plate. Each point represents one track. Curve A is $E_p = 2{\cdot}6 \cos^2\theta$ for the case of a point source at $\phi = 0$. Curves B and C are $E_p = 2{\cdot}6 \cos^2\theta$ for point source at $\phi = 0$. Curves B and C are $E_p = 2{\cdot}6 \cos^2\theta$ for point sources at $\phi = \pm 30°$. ϕ and θ are illustrated in Fig. 10.

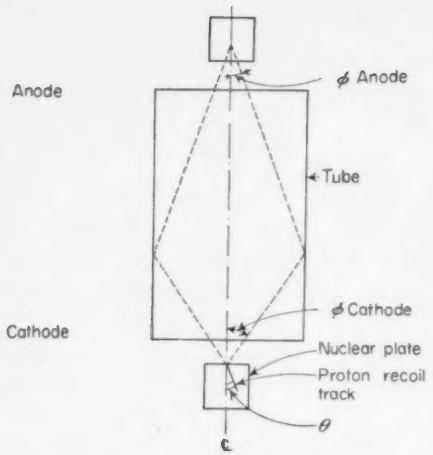

FIG. 10.—Sketch illustrating angles θ and ϕ. θ is the angle of recoil in the emulsion plane; ϕ is the angle the neutron source subtends at the plate. In the present experiment it is necessary to postulate ϕ up to $\pm 30^\circ$ at the cathode and up to $\pm 15^\circ$ at the anode.

plate a significant number of recoil protons had energies between 2·45 and 2·6 MeV. The maximum proton energy observed was $2{\cdot}61 \pm 0{\cdot}025$ MeV, which established that some of the neutrons must have had energies extending at least up to this value. Such neutrons could have arisen from a reaction centre of mass moving from anode to cathode with a velocity equivalent to 20 keV and impinging on a stationary target. On the anode plate most of the recoil protons had energies less than 2·3 MeV, which was consistent with the above process: some tracks however were from recoil protons of energy between 2·3 and 2·45 MeV, and these could have arisen from a smaller velocity of the reacting system.

The poor statistics made it impossible to estimate what fraction of the yield arose from an acceleration process equivalent to the bombardment of a stationary target by 20 keV deuterons, but there is no reason to suppose that most of the neutrons were not produced in this way.

(6) *An investigation of the X-ray emission*

Ilford G films, placed under the discharge tube and shielded by various absorbing thicknesses of aluminium and tin, were used to investigate the energy of components of the radiation stopped by the walls. The optical densities under the stepped absorbers were compared with those exposed to radiation from an X-ray tube in a control experiment; it was found that the energy of the radiation corresponded closely to the applied voltage across the tube, i.e. 18 to 20 kV. The total flux per discharge at this energy was estimated to be 2×10^{10} photons. This radiation was probably due to electrons accelerated in the initial stages of the discharge by the applied voltage as found by PODGORNYI (1956).

Further measurements with counters and film badges outside the tube showed that there were less than 10^4 quanta of energy ~50 keV emitted per discharge; this is in contrast to the findings of LUKYANOV and PODGORNYI (1956) who observed yields of 5×10^{10} quanta per discharge of mean energy ~200 keV, accompanied by neutron yields $\sim 10^8$.

(7) *Further properties of the neutron production*

In general it was found that the neutron emission was remarkably insensitive to the variation of external parameters. Only two ways of significantly changing the principal features of the yield have been observed. When one of the plane electrodes was replaced by a ring electrode the total yield was increased, but was more erratic from shot to shot and more sensitive to H_z. Reducing the capacity of the bank by a factor of 5 to 10 lowered the yield, but made it much more sensitive to H_z and the scatter was also greater. A similar result was observed on the small bank when extra inductance was added (CURRAN *et al.*, 1958), and was probably due to the same effect, because reducing the capacity also increased the bank inductance: at $\frac{1}{8}$ bank capacity (12·5 μF) the total bank inductance was 35×10^{-9} H compared to 5×10^{-9} H at full capacity.

4. DISCUSSION

(1) *Discussion of the results*

By comparing the present results on the time of occurrence and the duration of the neutron pulse with those of other workers (Section 3(1)) it is seen that if $\dot{I}$ is increased from $\leqslant 10^{11}$ to $\sim 5 \times 10^{11}$ A/sec the neutron pulse (at the second or third pinch) becomes longer. If $\dot{I}$ is further raised to $\geqslant 10^{12}$ A/sec neutrons are produced at the first pinch and the neutron pulse does not have the characteristic fast rise-time and short duration associated with an origin from an $m = 0$ instability, as found by, e.g. ANDERSON *et al.* (1958) using $\dot{I} \sim 10^{11}$ A/sec.

The comparative insensitivity of the yield to H_z suggests that the neutron production in the present experiments and that from other high-power experiments (MATHER and WILLIAMS, 1958; OHLIN and SIEGBAHN, 1958) is different from that in the earlier experiments of ARTSIMOVICH *et al.* (1956), ANDERSON *et al.* (1958) and DUNAWAY and PHILLIPS (1958). Furthermore, in the present experiments the time and duration of the emission and the dependence of the yield on voltage and current are all similar with and

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without H_z, and it is concluded that the neutron producing mechanism is probably the same with or without H_z, and cannot therefore be associated with narrow necks in the plasma column.

The observation that the yield was insensitive to bank voltage and current suggests a non-thermonuclear origin and also that the temperature predicted by the pressure balance ($T \propto I^2$) was not achieved. From this it follows that the approximate agreement in Fig. 8 between the experimental points and the theoretical thermonuclear neutron yield (a result in contrast to many other experiments, in which the observed yield exceeded the theoretical thermonuclear by $\sim$100) does not imply a thermonuclear origin.

The nuclear plate experiment established an anisotropy in the neutron energy, which makes a thermonuclear origin very unlikely; the non-uniform production along the tube is also unexpected on a thermonuclear basis. The wide neutron source, together with the fact that most of the neutrons were emitted after the first maximum compression when the pinch had expanded, suggests that the origin of the neutrons may have been localized in the sheath.

It is apparent, therefore, that the neutron emission observed in the present experiments (and probably in other experiments with $\dot{I} \sim 10^{12}$ A/sec) is not thermonuclear; it is different from the emission observed with $\dot{I} \sim 10^{11}$ A/sec and cannot be accounted for by the Colgate $m = 0$ instability process (see ANDERSON *et al.*, 1958). In the following section possible acceleration mechanisms are considered, and reasons to account for the failure to observe a large thermonuclear yield are given.

(2) *The origin of the neutrons*

Acceleration processes. In the $m = 0$ mechanism proposed by Colgate the reacting deuterons are accelerated by large voltages ($V = I\dot{L}$) arising from the rapid growth of instability necks in the plasma column. The accelerating voltage in Colgate's experiment was measured to be 50 kV, and to account for this at the observed current of 150 kA it was necessary to postulate that the $m = 0$ necks contracted to 0·05 cm. This explained

(a) the sensitivity of the yield to axial fields of $\sim$100 gauss,

(b) the production of high voltage transients,

(c) the brief duration of the neutron emission.

However, at higher currents (e.g. $I \sim 300$ kA) large inductive voltages can develop along the plasma or parts of it due to radial pulsations or local contractions without the need for narrow necks (see Appendix B). Whereas in the former case the neutron production is intimately associated with the catastrophic growth of $m = 0$ instabilities, in the latter case this is not so, and sufficient voltages ($\sim$20 kV) could develop across stable oscillations in discharges with H_z. It is suggested that neutrons from many linear pinch experiments arise from a process in which deuterons are accelerated by $I\dot{L}$ voltages not necessarily associated with instabilities.

In some conditions, possibly associated with low rates of rise of current (e.g. 10^{11} A/sec; ANDERSON *et al.*, 1958; DUNAWAY and PHILLIPS, 1958) $m = 0$ instabilities cause the development of narrow necks leading to break-up of the pinch. The induced voltage results in a large anisotropy in the neutron energy together with the other features of a typical $m = 0$ instability process as described above. In other cases ($\dot{I} \sim 10^{12}$ A/sec), including the present experiments, the induced voltage develops at larger radii, the energy anisotropy is smaller and the yield is less. Because of the longer time-scale for the growth of the inductive voltage the rise time of the neutron pulse will be slower and the pulse duration longer. Furthermore, H_z fields need not have so marked an effect on the neutron yield but will reduce the $I\dot{L}$ voltage and hence the reacting deuteron energy as found by MATHER and WILLIAMS (1958).

It is not clear why, in the present experiments with $H_z = 0$, narrow $m = 0$ necks did not grow and result in large yields from the Colgate process. However, some experiments, such as the addition of inductance to the small bank at A.W.R.E. (CURRAN *et al.*, 1958), and the effects observed at reduced capacity or with a ring electrode in the present experiments, suggested that production showing many features of an $m = 0$ process can be induced.

In some experiments (MATHER and WILLIAMS, 1958) it was difficult to account for the enormous deuteron current required on the above process, which exceeded the total tube current when the neutron energy anisotropy was observed to be small. This could be explained by postulating collisions between deuterons moving towards the cathode and much slower deuterons moving towards the anode. σ_{DD} is such a steep function of deuteron energy at a few keV that collisions between oppositely directed particles could be the predominant process. Such a process will reduce the number of particles required to give the yield. For example, a 10 keV deuteron reacting with a 1 keV oppositely directed particle will give a shift equivalent to a 4·5 keV deuteron on a stationary target, and a reaction probability in the ratio σ (17 keV)/σ (4·5 keV) $\sim 10^4$. This mechanism requires that certain parts of the plasma move along the column with velocities of

about 10^7 cm/sec. Radial velocities of this magnitude have been observed from streak photographs in these experiments and the displacement of plasma axially in both directions by radial constrictions could give axial deuteron velocities of the same order. The above mechanism does not require the very high velocities of the process suggested by HAGERMAN and MATHER (1958) in which the yield is due to an intense thermonuclear reaction in a small volume of plasma moving down the column with a velocity approaching 10^8 cm/sec in the anode to cathode direction only.

ARTSIMOVICH *et al.* (1956) have postulated the acceleration of ions near the axis of the plasma where the magnetic field is low, and have measured large electric fields inside the plasma column. These electric fields will also accelerate electrons which will give large bursts of hard X-rays, which were observed in the Russian experiments. No hard X-rays were observed in the present work, so that it seems unlikely that this acceleration process can account for the neutron emission described here.

A further possible process is the FERMI (1954) acceleration of deuterons reflected several times from the imploding sheath. It can be shown, by plotting trajectories, that a particle will only make an appreciable energy gain in this manner if the sheath accelerates to a high radial velocity soon after leaving the walls. This is observed to occur when the initial dI/dt is very large, as in the present case, when deuteron energies from 10 to 15 keV can be predicted by this process. However, the charge exchange cross-sections (FITE, 1959) at energies $\lesssim$15 keV are large, and it is doubtful if sufficient deuterons to give a detectable yield can make the required number of reflections before they escape as neutrals. Diametral collisions of the ions in the sheath do not contribute to the emission at this time since the directed energy (deduced from streak photographs) is too small (300 eV at 50 μ Hg).

The thermonuclear process. Finally, we must consider why thermonuclear temperatures have not been achieved. Possible reasons are listed below:

(a) A significant fraction of the total current may flow outside the pinch at or near the walls.

(b) Radiation losses may cool the plasma. These losses could be due, for example, to optical transitions of partially stripped impurity atoms (POST, 1958) or microscopic instabilities in the pinch as discussed by TUCK (1958).

(c) The number of electrons contributing to a pressure balance in the pinch may be greater than for a pure deuterium plasma due to stripped impurity atoms drawn in in the early stages.

(d) The pinch may break up or be carried to the walls by instabilities.

(e) Impurities boiled off the walls may flood into the pinched plasma column.

(f) The high-energy tail of the ion Maxwell distribution may have insufficient time to grow before the current falls, or the plasma is cooled due to (b), (d) or (e) above, or by heat loss to the electrodes. Fast particles in the tail of the Maxwell distribution will be lost to the ends first.

There is evidence in these experiments (BODIN *et al.*, 1959) that after the first pinch there is sufficient build-up of current near the walls to limit the amount of energy put into the gas. ANDRIANOV *et al.* (1958), studying high-current ($\sim 10^6$ A) Z-discharges using bank voltages up to 120 kV, found that about half the current always flowed at the walls. This is one important factor in limiting the plasma temperature. No evidence has been found in streak photographs for gross turbulence or instabilities, in particular for discharges with H_z. There is experimental support for cooling by a flux of impurities from the walls, but not until the later stages (after several μsec); this could therefore quench a thermonuclear yield but not prevent it, unless the ion Maxwell tail had not developed. The theoretical times in Appendix A indicate that the ion Maxwell tail may not develop fully. There is optical evidence that the amount of impurity drawn in with the pinch is too small ($\sim$1 per cent) to seriously affect the number density, even if fully stripped. The importance of energy loss by radiation has not been established, but from the X-ray experiments it can be concluded that any significant energy loss must lie in the wavelength range >2 Å.

CONCLUSION

The neutron emission with $\dot{I} \sim 10^{12}$ A/sec, although different from that at $\dot{I} \sim 10^{11}$ A/sec, and inconsistent with the Colgate $m = 0$ instability mechanism, is not thermonuclear. It may be accounted for by a process in which deuterons are accelerated by inductive voltages due to radial motion of the plasma column. Large yields of thermonuclear neutrons were not observed, and of the reasons suggested to account for this, build-up of current at the walls or radiation cooling appear to be the most likely.

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APPENDIX A

Relaxation times

Because of the uncertainty in the values of number density and temperature in the pinch only rough values can be obtained, and these will be determined for a typical final number density of $n = 10^{17}$ ions/cm^3. This is for a starting deuterium gas pressure of 50 μ and a compression $\sim$30, which, within a factor 2, covers most of the relevant cases. The times, which are inversely proportional to n, are determined for several temperatures; the ion temperature in the pinch has been estimated indirectly by a number of methods (see BODIN, *et al.*, 1959) and is likely to be in the region of some hundreds of eV.

(1) *Ion thermalization.* The bulk of the ion energy distribution will approach Maxwellian in about the time, t_{ii}, for one large angle scatter due to distant collisions. An approximate value of t_{ii} is given by POST (1957) as follows:

$$t_{ii} = \frac{6 \times 10^{10} W^{3/2}}{n} \text{ sec} \qquad \text{(A1)}$$

where W is the ion energy in keV.

For $n_i = 10^{17}$ the value of t_{ii} at different mean ion energies is $t_{ii} = 0{\cdot}6\ W^{3/2}$ μsec. Typical values are as follows:

W (eV)	250	500	1000
t_{ii} (μsec)	0·075	0·21	0·6

The main part of the ion distribution will therefore become Maxwellian very rapidly just after the first pinch. The electrons thermalize in a time $t_{ee} \sim 1/60\ t_{ii}$ so will become Maxwellian almost at once. Thus under most conditions one can consider ion and electron temperatures.

(2) *The time, t_M, for the high energy tail of the Maxwellian ion distribution to grow.* Almost all the thermonuclear reactions at the likely pinch temperatures come from particles in the tail of the Maxwellian distribution. It can be shown* that at low temperatures most of the reactions come from particles with an energy W_M given by

$$W_M/T = 2/T^{1/3} \qquad W_M, \quad T \text{ in keV} \qquad \text{(A2)}$$

It can be shown also (see for example MACDONALD *et al.*, 1957) that the time, t_M, for the Maxwellian tail to grow to an energy W_M is of the order of one large angle collision period for a particle with an energy W_M.

* POST (1956) gives an expression for the energy W at which most reactions occur as $W/T = 8{\cdot}1/T^{1/3}$. However, as pointed out by COLGATE (1959) and others, at low temperatures reactions come from collisions between oppositely directed particles, and the tail need only grow to $W_M = W/4$, from which equation A2 follows.

Values of W_M given by (1) and the corresponding values of t_M (t_{ii} at W_M) are shown below.

T (eV)	250	500	1000
W_M (keV)	0·8	1·26	2·0
t_M (μsec)	0·4	0·85	1·7

A general point emerges from the above considerations which is very important in any discharge where the Maxwell tail does not have time to grow completely, due, e.g., to the onset of instabilities, some cooling process or insufficient energy in the bank, etc. The thermonuclear yield can increase much less rapidly with ion temperature, T_i, than would be expected on the assumption that the tail has formed. This arises because the time for the tail to grow to the appropriate values of W_M increases linearly with temperature.

Also, in any discharge with rapid heating, the thermonuclear neutron output at low temperatures need not rise rapidly with time; its rise will be predominantly determined by the growth of the Maxwell tail and can be relatively slow.

(3) *Equipartition of energy between ions and electrons.* At the time of the first pinch the ions will have a higher energy than the electrons, by a factor $m_i/m_e \sim 3600$. The hot ions will loose energy by heating electrons and T_e will exponentially approach T_i with a time constant τ_e. This energy transfer will occur as well as any subsequent ion and electron heating which may take place, such as isentropic compression.

The equipartition time is given approximately by POST (1956) as

$$\tau_e = \frac{1{\cdot}1 \times 10^{12} T_e^{3/2}}{n} \text{ sec}, \quad T_e \text{ in keV} \qquad \text{(A3)}$$

$$n = 10^{17}, \quad \tau_e = 1{\cdot}1 T_e^{3/2} \ \mu\text{sec}.$$

To obtain the order of magnitude of the time for the ion and electron temperatures to approach one another the value of T_e (above) is put equal to $\frac{1}{2}T_i$, as, apart from further heating, T_e and T_i both tend to $\frac{1}{2}T_i$ eventually. Values of τ_e are as follows.

T_i(eV)	250	500	1000
τ_e(μsec)	0·5	1·4	3

(4) *Pressure balance.* The pressure balance may be written (for deuterium)

$$I^2 = 2Nk(T_e + T_i)f$$

where N is the line density of ions or of electrons in the tube and f is the fraction of gas swept up in the pinch ($0 < f < 1$).

If

$$T_e = \alpha T_i$$
$$I^2 = 2NkT_i f(1 + \alpha)$$

$\alpha = 1$ if τ_e is small compared to the length of the neutron burst, t_n, and $\alpha = 0$ if $\tau_e > t_n$. The table in Appendix A(3) shows that $\tau_e \sim t_n$, and so $1 > \alpha > 0$, i.e. there is some pressure contribution from nkT_e.

The value of f has not been measured, but taking $f = 1$ gives a lower limit of the temperature.

Values of T_i have been determined for a wide range of values of f and α, as in the analysis of DUNAWAY and PHILLIPS (1958).

In Fig. 8 the neutron yields determined from the pressure balance temperature are shown for the values of T_i obtained in the following cases only.

(a) $f(1 + \alpha) = 2$ i.e. $f = 1$, $\alpha = 1$ (lower limit of temperature)

(b) $f(1 + \alpha) = 1$ e.g. $f = 1$, $\alpha = 0$
or $f = \frac{1}{2}$ $\alpha = 1$ etc.

(c) $f(1 + \alpha) = \frac{1}{2}$ e.g. $f = \frac{1}{2}$ $\alpha = 0$ etc.

APPENDIX B

The growth of inductive voltages along the plasma

The voltage developed along the plasma is

$$V = d/dt\,(LI) \cong I\dot{L}$$

when the inductance, L, changes rapidly.

For a Z-pinch

$$L = 2 \log R/r \times 10^{-9} \text{ H/cm}$$
$$\dot{L} = -2\dot{r}/r \times 10^{-9} \text{ H/sec/cm}$$
$$\therefore \qquad V = -2\,I\dot{r}/r \times 10^{-9} \text{ kV/cm}$$

I is in kA. r will be dependent on I in a way determined by the type of contraction. Values of V for possible values of $\dot{r}$ and r are given in the table for $I = 300$ kA. V is in kV/cm.

r (cm) \ $\dot{r}$ (cm/sec)	5×10^6	10^7	2×10^7
0·1	30	60	120
0·5	6	12	24
1	3	6	12
2	1·5	3	6

Thus, at currents of several hundred kA large voltages can develop at values of r and $\dot{r}$ of the order of those existing in the present discharge.

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CRITICAL CURRENT FOR BURN-OUT IN AN OGRA-TYPE DEVICE

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Abstract—A complete algebraic analysis has been obtained for the variation of the steady-state ion density n_+ with injected current I in an OGRA-type fusion device (i.e. a device based on trapping of ions by break-up of energetic molecular ions on collision with either the background gas or trapped ions). The most general variation of n_+ with I is shown to be an S-curve with at most three roots of n_+ for a given input I. A physical interpretation of these three roots is given. In addition algebraic expressions are obtained for the two currents at which the bends in the S-curve occur. It will be necessary to attain the larger current in order to build up a high-density plasma when the density is being increased from below. On the other hand, once the high density has been achieved, it may be maintained by steady injection of a current larger than the lower value. Parameters corresponding roughly to the specifications of OGRA are used to obtain some numerical results. In the final appendix, the previously published formulas for burn-out in DCX are extended to include effects of neutral backstreaming from the input beam and 'ion-pumping.'

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1. STEADY-STATE RELATION BETWEEN ION DENSITY AND INJECTED CURRENT

IN two previous publications (SIMON 1958 and 1959) an expression was derived for the critical current at which formation of a plasma by high-energy injection will begin. This previous expression was for a case in which the trapping mechanism (although not specified in detail) was localized and did not depend on either the neutral gas in the device, the trapped ion density or the dimensions of the system. A trapping mechanism of this sort is provided by the arc in DCX (BARNETT 1958a).

The situation is quite different in a proposed fusion device such as OGRA (KURCHATOV 1958). Here the injected molecular ions have a long mean-free-path L before they strike the injector snout and trapping occurs by virtue of the dissociation of the molecule on collision with either the background gas (cross-section $\sigma_B{}^0$), the trapped ions ($\sigma_B{}^+$), or other molecular ions in transit ($\sigma_B{}^{2+}$). Nevertheless, one might suspect on physical grounds that a critical current also exists in this case and indeed such an expression has been found. The result is somewhat more complex than in the case of DCX because of a feed-back which is inherent in the gas-break-up scheme. The onset of neutral burn-out results in a reduction of the neutral break-up centres as well as an increase of the ion break-up centres and hence has a back effect on the input trapped current.

A complete algebraic analysis of the steady-state equations has been achieved in the case when one can neglect the contribution of the molecular ions to burn-out or to break-up of other molecules as compared to the effect of the trapped ions and the neutral gas. (This is a highly valid approximation in almost all cases of interest.) The total mean-free-path λ of the injected molecular ions is then:

$$\frac{1}{\lambda} = \frac{1}{L} + \hat{N}_0\sigma_B{}^0 + n_+\sigma_B{}^+ \tag{1}$$

where $\hat{N}_0$ is the average neutral density (atomic) external to the plasma region and n_+ is the trapped energetic ion density. (It has been assumed that the slow ions resulting from ionization of the neutrals in the plasma region contribute equally to break-up as do the neutrals themselves. The sum of the slow ion and neutral densities in the plasma interior should remain equal to the external neutral density even after burn-out.) The probability that a molecule will break up after a path length x is then

$$p(x) = e^{-x/\lambda}\frac{dx}{\lambda_B} \tag{2}$$

where

$$\frac{1}{\lambda_B} = \hat{N}_0\sigma_B{}^0 + n_+\sigma_B{}^+ \tag{3}$$

The percentage trapping is found by integrating equation (2) over all space. The result is

$$\text{percentage B.U.} = \frac{\lambda}{\lambda_B} = \frac{\hat{N}_0\sigma_B{}^0L + n_+\sigma_B{}^+L}{1 + \hat{N}_0\sigma_B{}^0L + n_+\sigma_B{}^+L} \tag{4}$$

The steady-state rate of trapping of energetic ions is

* Operated by Union Carbide Corporation for the U.S. Atomic Energy Commission.

obtained by multiplying equation (4) by I, where I is the injected (number) current of molecular ions. It is assumed that trapping by dissociation of the molecular ion will dominate over trapping by actual ionization.

The trapped ions will be lost from the plasma in two ways. Either they undergo a charge exchange collision with a background gas atom or they scatter on each other and go out the mirrors. Since mirror loss will be negligible until well after burn-out sets in, we ignore any contribution to mirror loss due to scattering of the ions against the neutral gas atoms. The average neutral density $\bar{n}_0$ in the plasma interior is simply related to the exterior neutral density $\hat{N}_0$ (SIMON 1959).

$$\bar{n}_0 \cong \frac{\hat{N}_0}{1 + \frac{\bar{l}}{\lambda}} \tag{5}$$

$$= \frac{\hat{N}_0}{1 + \frac{\bar{l}v}{v_0} n_+ \sigma_d{}^+} . \tag{6}$$

Here $\bar{l}$ is the mean chord length of the plasma volume and λ is the mean-free-path for destruction of a gas atom (by either ionization or charge exchange) as it enters the plasma. Equation (6) expresses this mean-free-path in terms of the properties of the ions. The quantity $\sigma_d{}^+$ represents the sum of the cross-sections for ionization and charge exchange of an ion incident on a gas atom:

$$\sigma_d{}^+ = \sigma_i + \sigma_{cx}. \tag{7}$$

The background gas has been assumed to be of the same species as the injected molecular ion (i.e. deuterium or hydrogen in all cases of interest). Hence a single ionization or charge exchange suffices to make it impotent. The average velocity of the gas atom is denoted by v_0 and v is the ion velocity. Throughout this paper we assume that energy degradation processes are unimportant and that the ions remain at their initial energy until they are lost.

The steady-state loss rate of ions by charge exchange follows immediately from equation (6). It is:

$$= \frac{\hat{N}_0 n_+ \sigma_{cx} v V}{1 + \frac{\bar{l}v}{v_0} n_+ \sigma_d{}^+} \tag{8}$$

where V is the plasma volume. Finally, the mirror loss rate is assumed to be given by the usual binary expression (POST 1958),

$$= n_+{}^2 \sigma_c v P V \tag{9}$$

where σ_c is the 'effective 90-degree Coulomb scattering cross-section' by cumulative small angle scattering and P is the probability of an ion being in the loss cone after such a scattering. Combining equations (4), (8) and (9), we have the following steady-state equation for the ion density:

$$\frac{I}{V} \frac{\hat{N}_0 \sigma_B{}^0 L + n_+ \sigma_B{}^+ L}{1 + \hat{N}_0 \sigma_B{}^0 L + n_+ \sigma_B{}^+ L} = \frac{\hat{N}_0 n_+ \sigma_{cx} v}{1 + \frac{\bar{l}v}{v_0} n_+ \sigma_d{}^+} + n_+{}^2 \sigma_c v P. \tag{10}$$

An expression for $\hat{N}_0$, the neutral density exterior to the plasma, may be readily obtained by accounting for all the sources of gas. The background gas is assumed to come from a steady outgassing of the walls of the container and from backstreaming of neutrals resulting from the energetic ions and molecular ions which are injected and which finally strike the walls. We assume that a certain fraction Γ of the input molecular ion ultimately enter the vacuum region as neutral atoms. Of course this is a rather crude accounting and does not distinguish between the different re-entry probabilities of energetic ions and unbroken-up molecular ions which strike the wall. By conservation of atoms, we see that $\Gamma \leq 2$. The steady outgassing rate of the surface is denoted by $\alpha \bar{S}$ where $\bar{S}$ is the surface area of interest and α is the outgassing rate per unit area.

This influx of gas is balanced in two ways. First, neutrals are pumped off directly by external pumps acting on the system. If θ denotes the pumping speed of the external pumps, this loss rate is $\theta \hat{N}_0$. The second loss is due to the 'ion-pumping' action of the trapped plasma. Every neutral atom which enters the plasma volume and undergoes either an ionization or a charge exchange becomes a slow ion. This ion is trapped on a field line and rapidly moves out to the mirror regions. A large fraction of these slow ions may be permanently trapped and prevented from returning to the vacuum region, for instance by accelerating them into a titanium plate and continually evaporating new layers of titanium on that region. This is one possible form of 'ion-pumping.' Let σ denote the fractional probability that a slow ion does *not* return as a neutral. The rate of loss of neutrals by this mechanism is then:

$$= \sigma \frac{\hat{N}_0 n_+ \sigma_d{}^+ v}{1 + \frac{\bar{l}v}{v_0} n_+ \sigma_d{}^+} V.$$

The resultant steady-state equation for $\hat{N}_0$ is then:

$$\Gamma I + \alpha \bar{S} = \theta \hat{N}_0 + \frac{\sigma \hat{N}_0 n_+ \sigma_d{}^+ v V}{1 + \frac{\bar{l}v}{v_0} n_+ \sigma_d{}^+} . \tag{11}$$

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Now initially, in the absence of any input beam, the balance equation was

$$\alpha S = \theta N_0 \tag{12}$$

where N_0 is the initial neutral density in the vacuum region before injection began. Substituting equation (12) in equation (11), we have

$$\hat{N}_0 = \frac{\frac{\Gamma}{\theta} I + N_0}{1 + \frac{\sigma}{\theta} \frac{n_+ \sigma_d{}^+ v V}{1 + \frac{\bar{l}v}{v_0} n_+ \sigma_d{}^+}} \tag{13}$$

Equations (10) and (13) combined constitute an implicit equation in the variables n_+ and I. (Some of these last considerations on backstreaming and ion pumping have not been included in previous publications (SIMON 1958 and 1959) on burn-out in DCX. This is done here in Appendix B.)

2. THE INJECTED CURRENT AS A FUNCTION OF THE ION DENSITY

We wish to determine the variation of ion density n_+ with changes in the injected current I. The gross features of the variation may be obtained by an algebraic analysis. Equations (10) and (13) combined have the general form:

$$f(n_+, I) = 0. \tag{14}$$

Let us first determine the possible roots of this equation for fixed n_+.

We shall rewrite equation (14) so as to explicitly display it as a polynomial in I. From equation (10) we have:

$$\frac{I}{V}(\hat{N}_0 \sigma_B{}^0 L + n_+ \sigma_B{}^+ L) = \left[\frac{\hat{N}_0 n_+ \sigma_{ex} v}{1 + \frac{\bar{l}v}{v_0} n_+ \sigma_d{}^+} + n_+{}^2 \sigma_e v P \right] \times [1 + \hat{N}_0 \sigma_B{}^0 L + n_+ \sigma_B{}^+ L] \tag{15}$$

which has the general form, by equation (13),

$$\frac{I}{V}(AI + B) = (CI + D)(1 + AI + B) \tag{16}$$

where A, B, C and D are positive definite quantities and are independent of I. Equation (16) may be written

$$I^2 \left[\frac{A}{V} - AC \right] + I \left[\frac{B}{V} - C(1 + B) - AD \right] - D(1 + B) = 0. \tag{17}$$

Two cases may be distinguished:

(a) $\frac{1}{V} > C$: In this case the product of the two roots is negative.

Hence there is one positive real root and one negative real root.

(b) $\frac{1}{V} < C$; The coefficient of I in equation (17) may be written as:

$$= B\left(\frac{1}{V} - C\right) - C - AD.$$

Hence, in this case, it is negative. We see that the product of the roots is positive but that the sum of the roots is negative. This can only occur if the two roots are either both negative real or form a complex pair. There is no positive real root.

Hence, we see that there can be at most one positive real value of I for a given n_+.

3. THE ION DENSITY AS A FUNCTION OF THE INJECTED CURRENT

Let us now rewrite equation (15) in order to display it as a polynomial in n_+. We define the quantity κ as,

$$\kappa = \frac{\bar{l}v}{v_0} \sigma_d{}^+ + \frac{\sigma}{\theta} \sigma_d{}^+ v V, \tag{18}$$

and multiply through by $(1 + \kappa n_+)^2$ in equation (15). The result is:

$$\frac{I}{V}\left[\left(\frac{\Gamma}{\theta} I + N_0\right)\left(1 + \frac{\bar{l}v}{v_0} \sigma_d{}^+ n_+\right)\sigma_B{}^0 L + n_+ \sigma_B{}^+ L \times (1 + \kappa n_+)\right](1 + \kappa n_+) = \left[\left(\frac{\Gamma}{\theta} I + N_0\right) \times n_+ \sigma_{ex} v + n_+{}^2 \sigma_e v P (1 + \kappa n_+)\right] \cdot \left[(1 + n_+ \sigma_B{}^+ L)(1 + \kappa n_+) + \left(\frac{\Gamma}{\theta} I + N_0\right) \times \sigma_B{}^0 L \left(1 + \frac{\bar{l}v}{v_0} n_+ \sigma_d{}^+\right)\right] \tag{19}$$

which has the general form :

$$A n_+{}^5 - B n_+{}^4 + C n_+{}^3 - D n_+{}^2 + E n_+ - F = 0 \tag{20}$$

Here

$$A = \sigma_e v P \kappa^2 \sigma_B{}^+ L$$

$$B = -\sigma_e v P \kappa \left[\sigma_B{}^+ L + \kappa + \left(\frac{\Gamma}{\theta} I + N_0\right) \sigma_B{}^0 L \frac{\bar{l}v}{v_0} \sigma_d{}^+\right]$$

$$F = \frac{I}{V}\left(\frac{\Gamma}{\theta} I + N_0\right) \sigma_B{}^0 L.$$

We see immediately that the product of the roots is positive and that the sum of the roots is negative. It

follows at once that there is either only one positive real root or three positive real roots for n_+ for a given value of I.

The results of Sections 2 and 3 enable us to sketch the general variation of n_+ with I and this is shown in Fig. 1. The curve has the general form of an S-curve.* It should be noted that the root analysis given above does not exclude the possibility that the S-curve behaviour will occur several times instead of just once as shown in Fig. 1. The physical interpretation of the S-curve given in the next section makes it clear, however, that an S-curve will occur only once if at all. In this regard, it should be noted that in some cases no S-curve appears and instead the curve increases monotonically to a limiting value of n_+. The criterion for the occurrence of this behaviour is given below [see equation (32)].

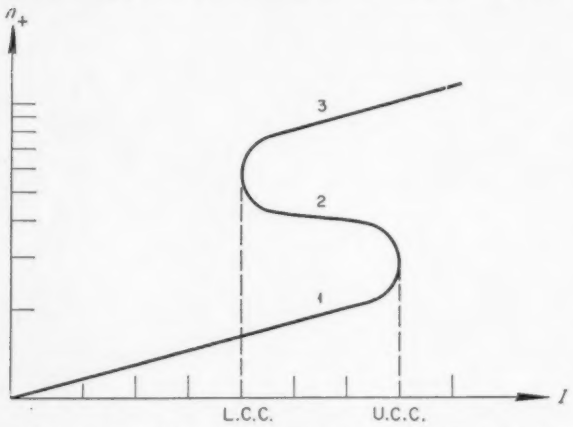

FIG. 1.—Variation of steady-state ion density with injected current for an OGRA-type device.

For comparison, the characteristic curve of n_+ versus I in the case of a fixed break-up device such as DCX is shown in Fig. 2.

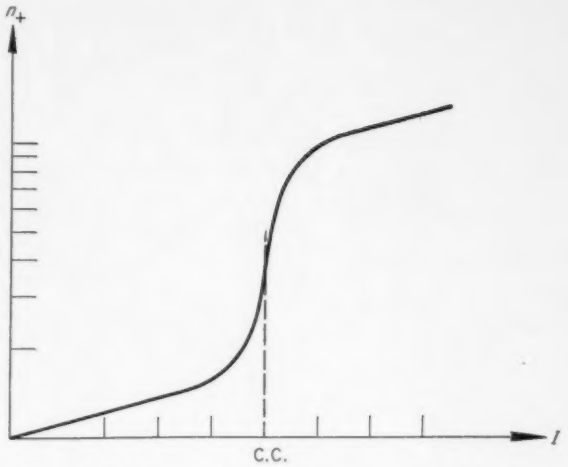

FIG. 2.—Variation of steady-state ion density with injected current for a DCX-type device.

4. PHYSICAL INTERPRETATION OF THE ROOTS

The multiple roots occurring in Fig. 1 have a straightforward physical interpretation. In region 1 neutral burn-out has set in. The steady-state solution is achieved by balance between charge exchange loss of the trapped ions and feed by break-up of the molecular ions on the neutral background (mirror loss is negligible). The second solution in region 2 corresponds to the point at which the ion density has risen and the neutral density has fallen such that the break-up on the ions now is the same as the previous break-up on the neutral gas. The charge exchange loss remains the same since it is proportional to the product $n_0 n_+$ (where n_0 is the average neutral density in the plasma region) and since, after burn-out, $n_0 \sim 1/n_+$. The final root of region 3 corresponds to the point at which mirror loss becomes more important than charge exchange loss.

The physical interpretation just given makes it clear that roots 1 and 3 are stable while root 2 is unstable. For example, suppose the system is in steady-state equilibrium at root 1 and that there is then a slight transient increase in the trapped ion density. The trapping rate will not increase appreciably since it is determined by the neutral gas density while the charge exchange loss rate increases linearly with n_+. Hence the ion density will decrease and this root is stable. If the system is in equilibrium at root 2, the reaction to this increase is quite different. The trapping rate increases linearly with n_+ since trapping is now primarily by collision with ions while the charge exchange loss rate (beyond burn-out) remains constant. Hence the ion density will continue to increase and the root is unstable. Finally consider the effect of a transient increase at root 3. The trapping rate increases linearly but the dominant mirror loss rate increases quadratically. Hence the ion density decreases and this root is stable.

5. THE UPPER CRITICAL CURRENT (U.C.C.)

Of course we can stop at this point and analyse the behaviour of any given device by obtaining a curve such as in Fig. 1 from equation (10) and (13) by numerical means. This is somewhat laborious, particularly if we wish to do a parameter survey and compare various proposals. Instead of the entire curve in Fig. 1, it would be almost as useful to know the two points at

* A recent paper by I. N. GOLOVIN (Harwell, April 1959, unpublished) states that KUZNETSOV and co-workers have numerical results indicating a behaviour of this sort.

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which the curve is vertical. These are denoted as the Upper Critical Current (U.C.C.) and Lower Critical Current (L.C.C.) in Fig. 1. It is clear that it is necessary to attain the U.C.C. in order to build up a high-density plasma when the density is being increased from below. On the other hand, once a high density has been achieved, it may be maintained by steady injection of a current larger than the L.C.C. A derivation of an approximate expression for the U.C.C. is given below and one for the L.C.C. is given in the next section.

Let us consider I as a function of n_+ and differentiate equation (14). We have:

$$\frac{\partial f}{\partial n_+} + \frac{dI}{dn_+}\frac{\partial f}{\partial I} = 0. \tag{21}$$

Now the U.C.C. and L.C.C. are distinguished by being the points at which $dI/dn_+ = 0$. Hence we have

$$\frac{\partial f}{\partial n_+} = 0. \tag{22}$$

The U.C.C. and L.C.C. are determined by the simultaneous solution of equation (14) and (22). Substituting in equation (22) from equation (10), we have

$$\begin{aligned}\frac{\partial}{\partial n_+}\left[\frac{I}{V}\frac{\hat{N}_0\sigma_B{}^0L + n_+\sigma_B{}^+L}{1+\hat{N}_0\sigma_B{}^0L + n_+\sigma_B{}^+L}\right] &= \frac{\partial}{\partial n_+}\left[\frac{\hat{N}_0 n_+\sigma_{cx}v}{1+\dfrac{\bar{l}v}{v_0}n_+\sigma_d{}^+} + n_+{}^2\sigma_e v P\right] \\ &= \frac{\partial}{\partial n_+}\left[\frac{\left(\dfrac{\Gamma}{\theta}I + N_0\right)n_+\sigma_{cx}v}{1+\kappa n_+} + n_+{}^2\sigma_e v P\right]\end{aligned} \tag{23}$$

where κ is defined in equation (18).

To get an expression for the U.C.C. we neglect the second term on the right side of equation (23). This corresponds to mirror loss and should be negligible according to the physical interpretation given earlier. To get a relatively simple result we must make one other approximation. We assume that the U.C.C. is well above burn-out, or in other words, that $\kappa n_+ \gg 1$ at that point. To first approximation, then, the bracket on the right-hand side of equation (23) is independent of n_+ and its partial derivative vanishes. Equation (23) becomes

$$\frac{\partial}{\partial n_+}\left[\frac{\hat{N}_0\sigma_B{}^0L + n_+\sigma_B{}^+L}{1+\hat{N}_0\sigma_B{}^0L + n_+\sigma_B{}^+L}\right] = 0 \tag{24}$$

Now

$$\frac{d}{dx}\left(\frac{y}{1+y}\right) = \frac{\dfrac{dy}{dx}}{(1+y)^2} \tag{25}$$

Hence equation (24) reduces to

$$\frac{\partial}{\partial n_+}\left(\hat{N}_0\sigma_B{}^0L + n_+\sigma_B{}^+L\right) = 0 \tag{26}$$

or

$$\frac{\partial}{\partial n_+}\left[\frac{\left(\dfrac{\Gamma}{\theta}I + N_0\right)\sigma_B{}^0L\left(1+\dfrac{\bar{l}v}{v_0}n_+\sigma_d{}^+\right)}{(1+\kappa n_+)} + n_+\sigma_B{}^+L\right] = 0 \tag{27}$$

After some algebra, this equation reduces to:

$$(1+\kappa n_+) = \sqrt{\frac{\left(\dfrac{\Gamma}{\theta}I + N_0\right)\sigma_B{}^0}{\sigma_B{}^+}\left(\frac{\sigma}{\theta}\right)\sigma_d{}^+vV} \equiv \phi \tag{28}$$

or

$$n_+ = \frac{\phi - 1}{\kappa} \tag{29}$$

We can now eliminate n_+ in equation (14) by use of equation (29). The result is an implicit equation in I, whose solution should be the U.C.C. This equation is easily found to be:

$$I = \frac{\left(\dfrac{\Gamma}{\theta}I + N_0\right)\sigma_{cx}vV\left(\dfrac{\phi-1}{\phi}\right)\left[1+\Lambda\sigma_B{}^0L + \left(\dfrac{\phi-1}{\kappa}\right)\sigma_B{}^+L\right]}{\kappa\left[\Lambda\sigma_B{}^0L + \left(\dfrac{\phi-1}{\kappa}\right)\sigma_B{}^+L\right]} \tag{30}$$

where

$$\Lambda = \left(\frac{\Gamma}{\theta}I + N_0\right)\left[1 + \left(\frac{\phi-1}{\kappa}\right)\frac{\bar{l}v}{v_0}\sigma_d{}^+\right]\frac{1}{\phi} \tag{31}$$

The solution of equation (30) gives us the U.C.C. and the corresponding value of n_+ is then given by equation (29).

It should be noted that we assumed initially that $\kappa n_+ \gg 1$ at the U.C.C. In order to be self-consistent, then, the value of κ corresponding to our solution

should be large compared to unity. Our numerical results seem to indicate that quite good answers are obtained for the U.C.C. even when ϕ is of the order of 2. When ϕ is less than unity, the approximation breaks down completely and in this case we have found no simple relation which can be used.

It has not yet been demonstrated that there is only one real positive value of I which satisfies equation (30). We have succeeded in showing this in the limit of $\phi \gg 1$. The method is similar to that used earlier and consists in expressing equation (30) as a (3rd order) polynomial in ϕ. It can be shown that one positive real solution for ϕ exists if

$$\frac{\theta \bar{l}}{V v_0} + \sigma > \Gamma \frac{\sigma_{cx}}{\sigma_d^+} \tag{32}$$

If the inequality is reversed, there is no positive real root. From the definition of ϕ, equation (28), we see that equation (32) is a necessary condition for the existence of an U.C.C.

One final point before leaving this section. If we make the opposite assumption to that made at the beginning of this section and take $\kappa n_+ \ll 1$, it is then easy to show that no simultaneous solution of equation (23) and (14) can exist. This verifies our physical understanding of the roots which requires that burn-out occur in order that an U.C.C. exist.

6. THE LOWER CRITICAL CURRENT (L.C.C.)

The L.C.C. is determined by the simultaneous solution of equation (23) and (14). Our previous physical picture of conditions at the L.C.C. indicated that the neutral background should be entirely negligible at that point. Hence we neglect the terms $\hat{N}_0 \sigma_B{}^0 L$ compared to $n_+ \sigma_B{}^+ L$ in equation (23). Correspondingly we assume that $n_+ \sigma_B{}^+ L \gg 1$ and $\kappa n_+ \gg 1$. Equation (23) becomes:

$$\frac{\partial}{\partial n_+}\left[\frac{I}{V}\left(1 - \frac{1}{n_+ \sigma_B{}^+ L}\right)\right] = \frac{\partial}{\partial n_+}\left[\frac{\left(\frac{\Gamma}{\theta} I + n_0\right)\sigma_{cx} v}{\kappa}\left(1 - \frac{1}{\kappa n_+}\right) + n_+{}^2 \sigma_e v P\right] \tag{33}$$

The solution of this equation is:

$$n_+{}^3 = \frac{1}{2\sigma_e v P}\left[\frac{I}{V \sigma_B{}^+ L} - \frac{\left(\frac{\Gamma}{\theta} I + N_0\right)\sigma_{cx} v}{\kappa^2}\right] \tag{34}$$

which can be rewritten by use of equation (14) as:

$$n_+ = \left\{\frac{1}{3\sigma_e v P}\left[\frac{I}{V} - \frac{\left(\frac{\Gamma}{\theta} I + N_0\right)\sigma_{cx} v}{\kappa}\right]\right\}^{\frac{1}{2}} \equiv \delta \tag{35}$$

Upon substitution in equation (14), we obtain an implicit equation in I,

$$I = \frac{\left(\frac{\Gamma}{\theta} I + N_0\right)\sigma_{cx} v V\left(1 - \frac{3}{2\kappa\delta}\right)}{\kappa\left(1 - \frac{3}{2\delta\sigma_B{}^+ L}\right)} \tag{36}$$

Just as in the case of the U.C.C., this expression is self-consistent only if the resulting solution is such that $\kappa\delta \gg 1$ and $\sigma_B{}^+ L\delta \gg 1$. Once again, in this limit, it is immediately demonstrated that there is at most one real positive root of equation (36) and that this root exists only if the inequality of equation (32) is satisfied.

7. NUMERICAL SOLUTIONS FOR 'OGRA'

A series of numerical runs was made using parameters which should be roughly that of the OGRA experiment. These parameters were:

$L = 10^5$ cm
$\bar{l} = 50$ cm
$V = 2{\cdot}35 \times 10^6$ cm^3
$\theta = 4 \times 10^7$ cm^3/sec
$P = 0{\cdot}388$.

Both $H_2{}^+$ and $D_2{}^+$ injection was considered, each at an energy of 200 keV. Furthermore, since there appear to be some large differences in published values, two sets of cross-sections were used. One set, hereafter denoted by USSR, was taken from the curves of Federenko as given in KURCHATOV (1958). The other set, hereafter denoted as USA, were based on BARNETT's results (BARNETT 1958b) and the Bethe formula for the ionization cross-section (MOTT 1949). These cross-sections (per gas atom) are tabulated below:

(a) Deuterium:

$v_0 = 1{\cdot}26 \times 10^5$ cm/sec
$v = 3{\cdot}12 \times 10^8$ cm/sec
$\sigma_e \cong 2{\cdot}6 \times 10^{-22}$ cm^2

	USA	USSR
$\sigma_B{}^0$	5×10^{-17}	$1{\cdot}65 \times 10^{-16}$
$\sigma_B{}^+$	5×10^{-17}	$1{\cdot}65 \times 10^{-16}$
σ_{cx}	9×10^{-17}	$7{\cdot}5 \times 10^{-17}$
$\sigma_d{}^+$	$3{\cdot}02 \times 10^{-16}$	$2{\cdot}6 \times 10^{-16}$

(b) Hydrogen:

$v_0 = 1{\cdot}78 \times 10^5$ cm/sec
$v = 4{\cdot}4 \times 10^8$ cm/sec
$\sigma_e = 2{\cdot}6 \times 10^{-22}$ cm^2

	USA	USSR
$\sigma_B{}^0$	5×10^{-17}	$1{\cdot}3 \times 10^{-16}$
$\sigma_B{}^+$	5×10^{-17}	$1{\cdot}3 \times 10^{-16}$
σ_{cx}	$1{\cdot}2 \times 10^{-17}$	3×10^{-18}
$\sigma_d{}^+$	$1{\cdot}32 \times 10^{-16}$	$1{\cdot}08 \times 10^{-16}$

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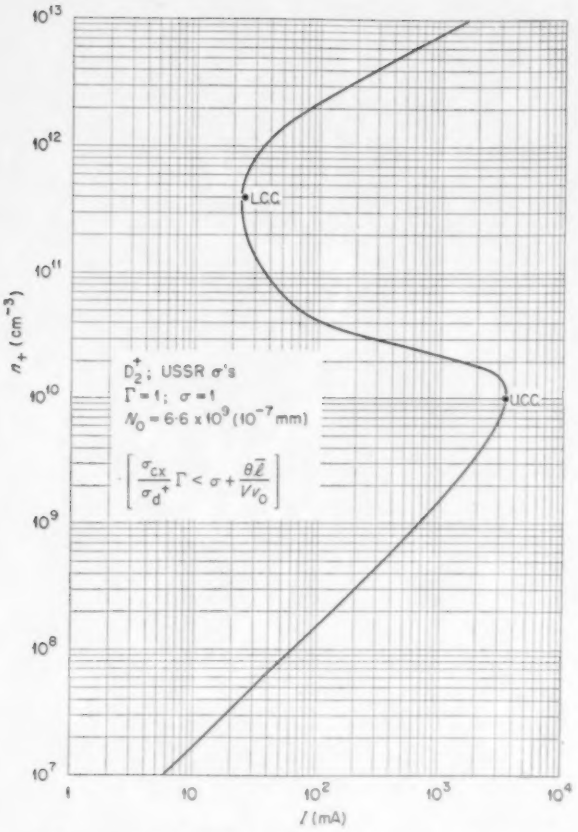

FIG. 3.—Variation of steady-state ion density with injected current.

A series of exact solutions of equations (10) and (13) were obtained and the results are plotted in Figs. 3–14. The Upper Critical Current and Lower Critical Current are calculated for several values of the initial pressure N_0 and for various values of σ and Γ. These results are given in Figs. 15–18. In addition, the calculated values of U.C.C. and L.C.C. are indicated for each case in Figs. 3–14.

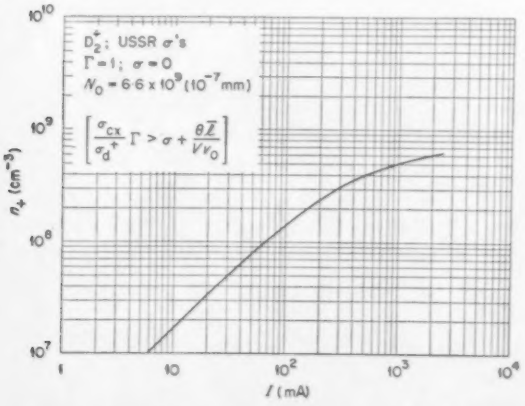

FIG. 4.—Variation of steady-state ion density with injected current.

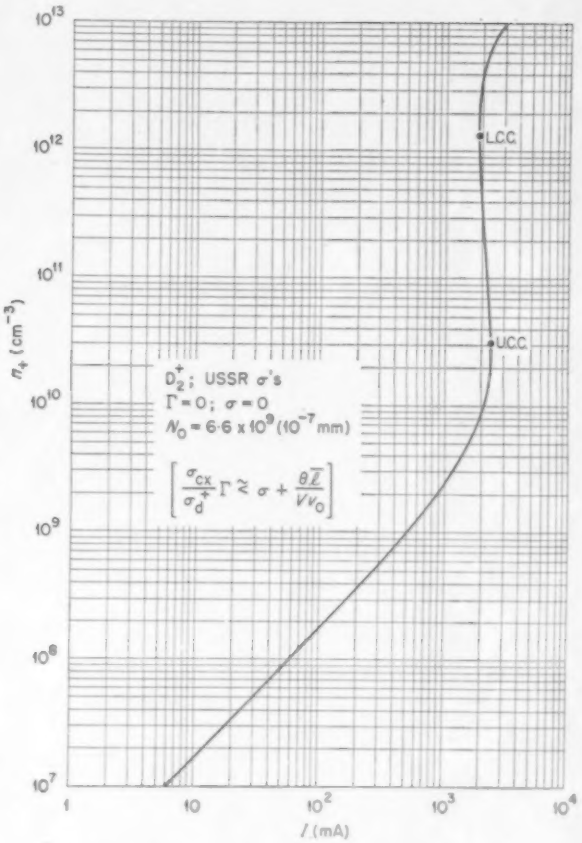

FIG. 5.—Variation of steady-state ion density with injected current.

8. SUMMARY

It is shown that the general shape of the curve which plots steady-state ion density versus injected current is an *S*-curve (see Fig. 1) for a gas break-up trapping device such as OGRA. The actual curve may be calculated by numerical means by use of equations (10) and (13). Approximate expressions are obtained for the two points at which the curve is vertical. The upper critical current (U.C.C.) is the solution of the implicit equation (30) and (31). The corresponding density is then calculated in equation (29). The lower critical current is the solution of the implicit equations (36) and (35). The corresponding density is then given by equation (35). Burn-out and the characteristic *S*-curve will only occur if the inequality of equation (32) is satisfied. If not, there is simply a monotonic increase of the density to a limiting value as the current increases indefinitely.

A physical interpretation of these roots is given in Section 4. A numerical survey of U.C.C. and L.C.C. for an experimental device having the dimensions of the Russian OGRA is represented in Figs. 3–18.

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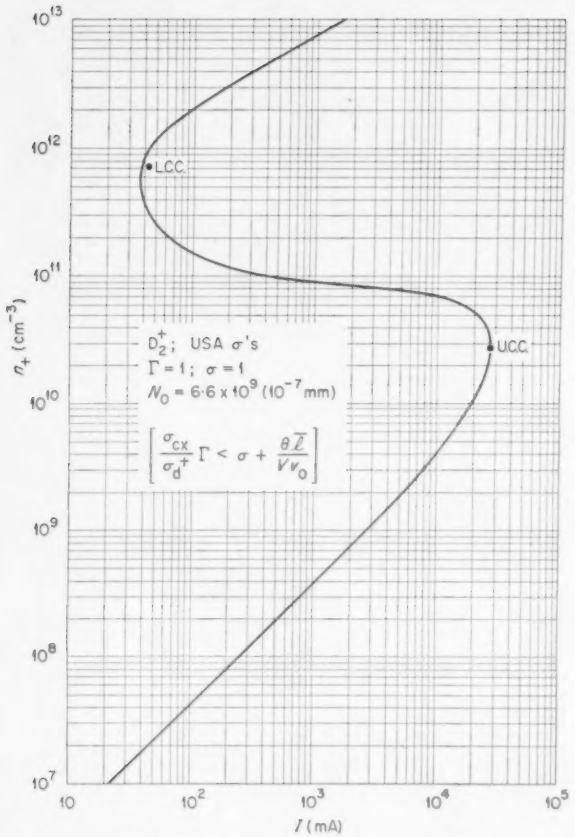

FIG. 6.—Variation of steady-state ion density with injected current.

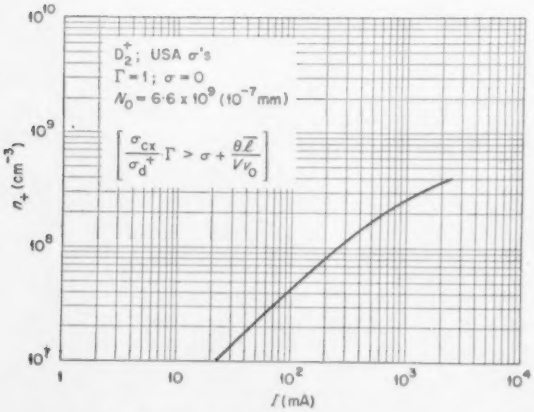

FIG. 7.—Variation of steady-state ion density with injected current.

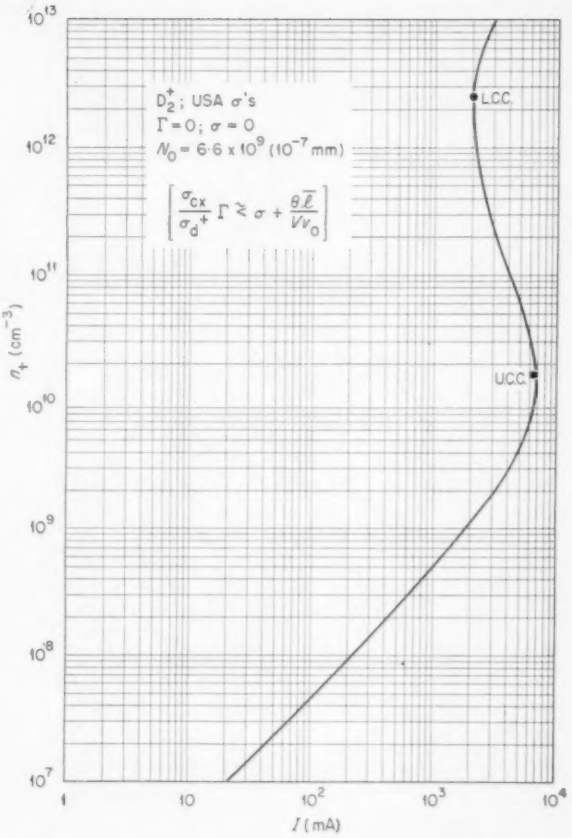

FIG. 8.—Variation of steady-state ion density with injected current.

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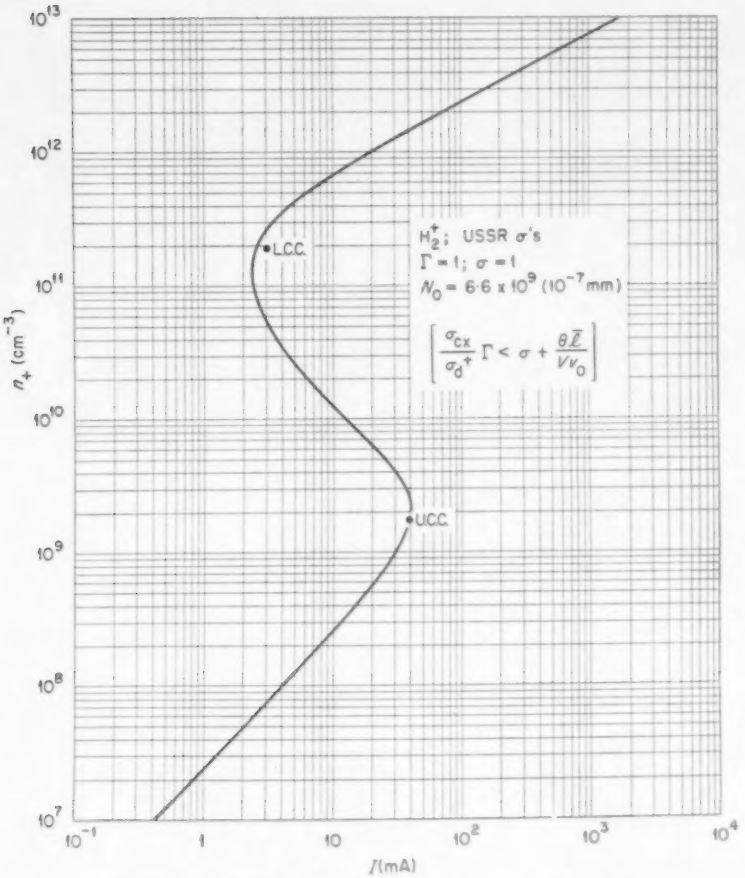

FIG. 9.—Variation of steady-state ion density with injected current.

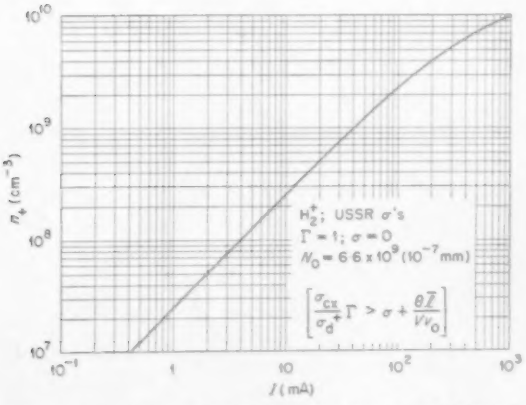

FIG. 10.—Variation of steady-state ion density with injected current.

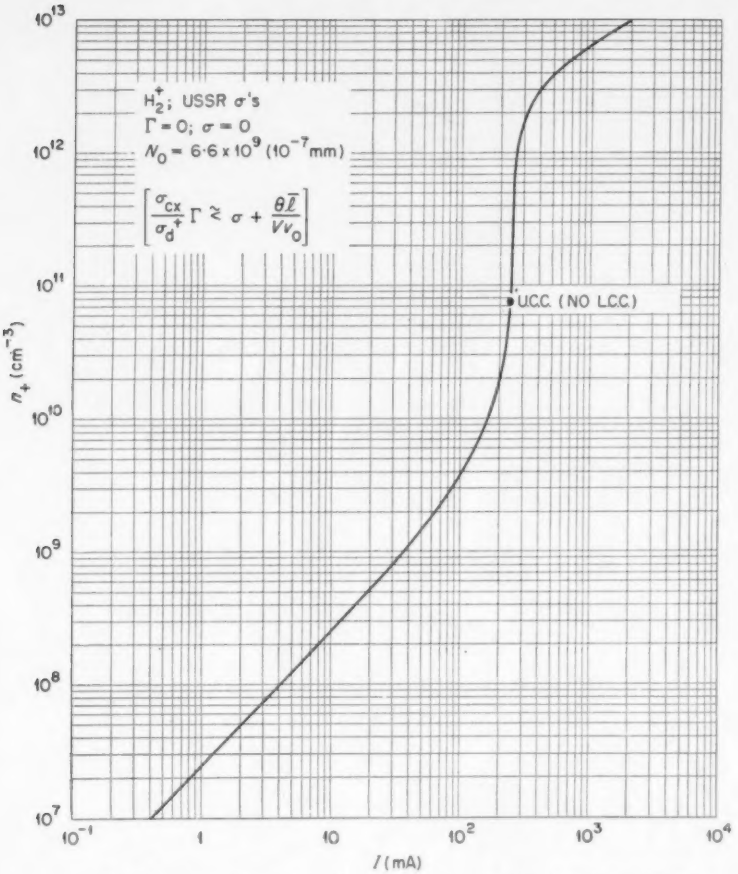

FIG. 11.—Variation of steady-state ion density with injected current.

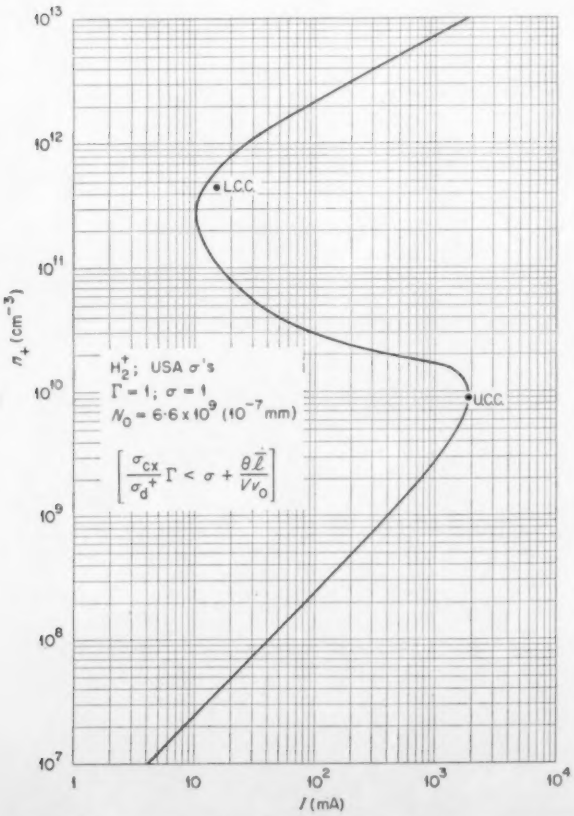

FIG. 12.—Variation of steady-state ion density with injected current.

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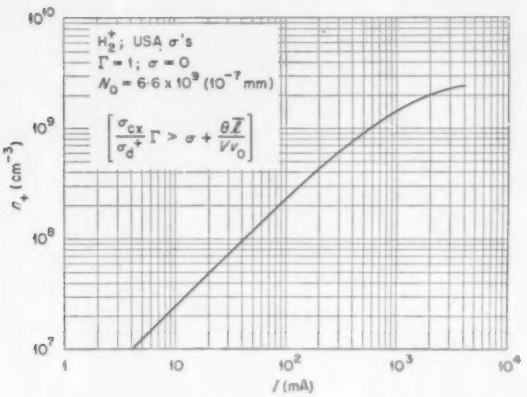

FIG. 13.—Variation of steady-state ion density with injected current.

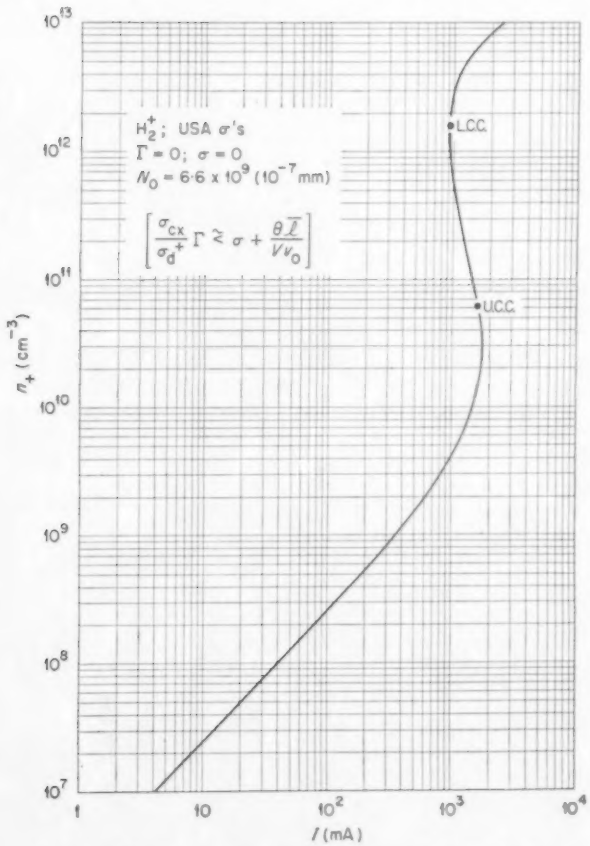

FIG. 14.—Variation of steady-state ion density with injected current.

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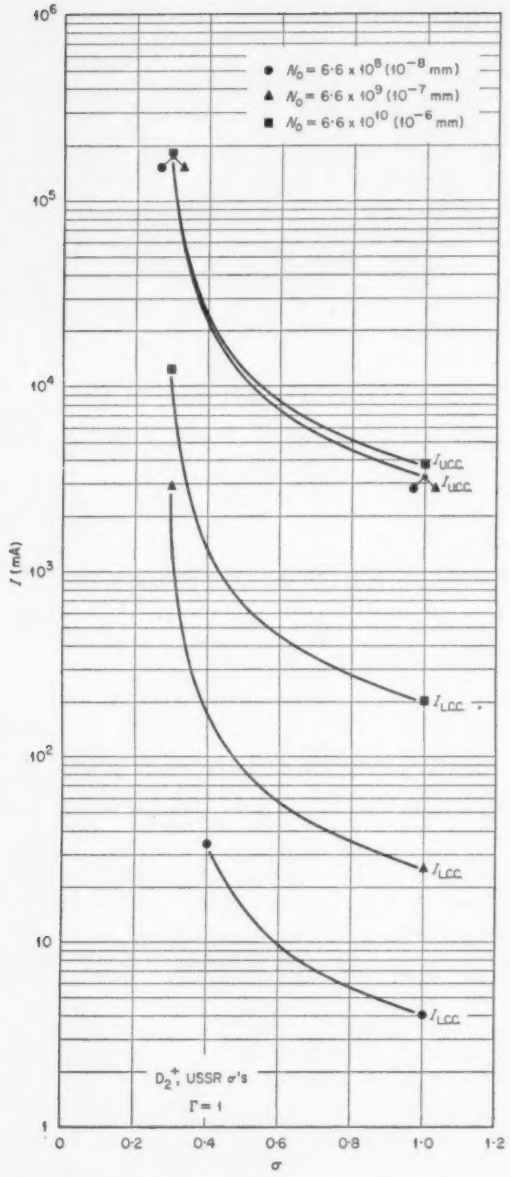

FIG. 15.—Variation of upper and lower critical currents with σ (the ion pumping efficiency).

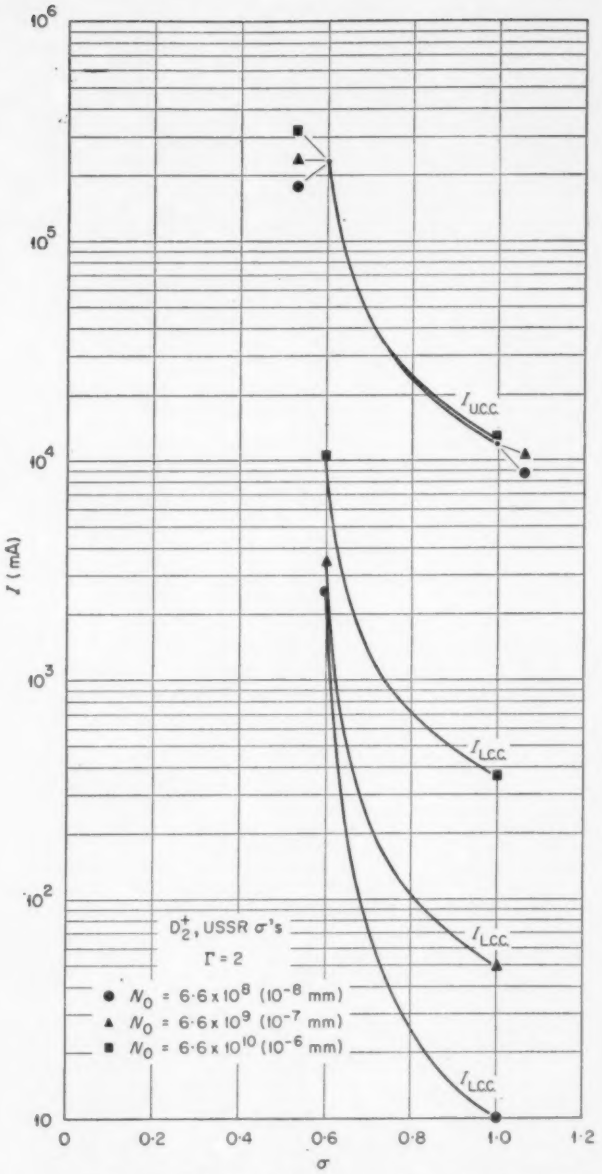

FIG. 16.—Variation of upper and lower critical currents with σ (the ion pumping efficiency).

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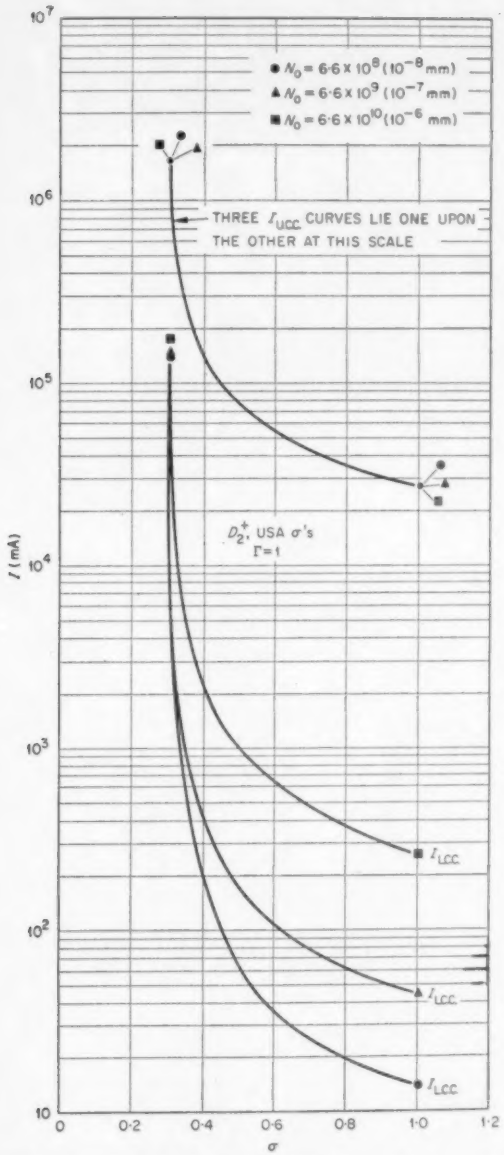

FIG. 17.—Variation of upper and lower critical currents with σ (the ion pumping efficiency).

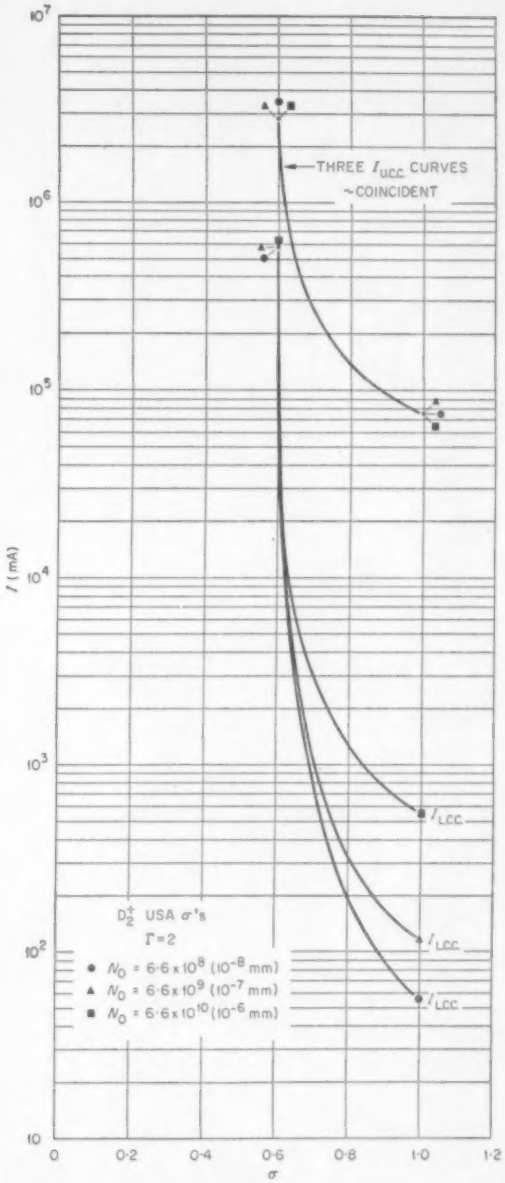

FIG. 18.—Variation of upper and lower critical currents with σ (the ion pumping efficiency).

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Appendix A considers the effect of treating the path length L as a fixed range rather than a mean-free-path and Appendix B applies the ion pumping consideration to previous formulas for burn-out in DCX.

Acknowledgements—I am greatly indebted to M. E. LAVERNE for his patient and careful handling of the numerical calculations. I am also most indebted to R. C. GILBERT and R. E. HESTER for calling my attention to the possibility of multiple roots in the gas break-up case.

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APPENDIX A

The path length L considered as a fixed range

It is not entirely clear whether the path length L should be considered to be a mean-free-path (as it is in the text) or a fixed range. If all injected particles entered at the same spatial point with the same angle then they would all ultimately strike some obstacle after traversing a fixed path length (in the absence of transient perturbations). More realistically, the injected beam has a rather wide spread in angle. The long path length and non-uniform magnetic field then produce a great deal of orbital precession and randomization and a statistical factor (i.e. a mean-free-path) would seem to be more correct. Actually the calculation below indicates that the difference is not important numerically, anyway.

Let us calculate the steady-state trapping rate if there is a fixed path length L. This is obviously

$$= 1 - \exp\left[-(\hat{N}_0\sigma_B{}^0 + n_+\sigma_B{}^+)L\right] \tag{A1}$$

which should be compared with equation (4). If we define

$$x = (\hat{N}_0\sigma_B{}^0 + n_+\sigma_B{}^+)L \tag{A2}$$

then the ratio of these two expressions is:

$$R(x) = \frac{(1 - e^{-x})(1 + x)}{x} \tag{A3}$$

This ratio approaches unity for x very small and for x very large compared to unity. The table below lists numerical values of R for several values of y, where $y = e^{-x}$.

$y(e \equiv {}^{-x})$	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
R	1	1·29	1·30	1·28	1·25	1·22	1·18	1·14	1·10	1·05	1

APPENDIX B

Extension of the DCX *burn-out formula to include ion-pumping*

An approximate expression for the critical current has been derived in SIMON (1958; 1959). We shall now derive it in a different way and then generalize to include ion-pumping. Let us define 'burn-out' to be that point at which the mean-free-path of a neutral entering the plasma becomes equal to the mean chord length of the plasma volume. In the notation of this report:

$$\lambda = \bar{l} \tag{B1}$$

or

$$\frac{v_0}{vn_+\sigma_d{}^+} = \bar{l} \tag{B2}$$

Now, since mirror loss is negligible at this point, the density is directly related to the input by the lifetime against charge exchange. Thus

$$n_+V = I\tau_{cx} \tag{B3}$$

$$= \frac{I}{\bar{n}_0\sigma_{cx}v} \tag{B4}$$

Upon substitution of equation (B4) in equation (B2), we have

$$I_{\text{crit.}} = \bar{n}_0\frac{\sigma_{cx}}{\sigma_d{}^+}\frac{v_0V}{\bar{l}} \tag{B5}$$

Now

$$\bar{l} = \frac{4V}{S} \tag{B6}$$

and, by SIMON (1959) equations (2) and (7),

$$\bar{n}_0 = \frac{\hat{N}_0}{1 + \dfrac{\bar{l}}{\lambda}} \tag{B7}$$

$$= \frac{\hat{N}_0}{2} \quad \text{at burn-out.} \tag{B8}$$

where $\hat{N}_0$ is the neutral density external to the plasma. Hence

$$I_{\text{crit.}} = \frac{\hat{N}_0}{2}\frac{v_0S}{4}\frac{\sigma_{cx}}{\sigma_d{}^+} \tag{B9}$$

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This differs by a factor of 2 from the result given in SIMON (1958). Of course, one cannot say which one is more 'correct.' Burn-out is not a precisely defined event and either approximate expression is a useful measure of its region of onset.

The external density, $\hat{N}_0$, is determined by the balance of wall out-gassing, ion pumping, external pumping, and backstreaming from the injected beam precisely as in Section 1 of the text. Hence, by equation (13), we have

$$\hat{N}_0 = \frac{\frac{\Gamma}{\theta} I + N_0}{1 + \frac{\sigma}{\theta} \frac{n_+ \sigma_d^+ v V}{1 + \frac{\bar{l}}{\lambda}}} \tag{B10}$$

At burn-out, equation (B1) and (B2) apply, which gives

$$\hat{N}_0 = \frac{\frac{\Gamma}{\theta} I + N_0}{1 + \frac{\sigma V v_0}{2\theta \bar{l}}} \tag{B11}$$

Upon substitution of equation (B11) in equation (B9) and rearranging, we find

$$I_{\text{crit.}} = \frac{N_0 \frac{v_0 S}{8} \frac{\sigma_{ex}^+}{\sigma_d^+}}{1 + \frac{S v_0}{8\theta} \left(\sigma - \Gamma \frac{\sigma_{ex}}{\sigma_d^+} \right)} \tag{B12}$$

It should be noted that burn-out will be possible only if

$$\frac{\sigma_{ex}}{\sigma_d^+} \Gamma < \sigma + 2 \frac{\theta \bar{l}}{V v_0} \tag{B13}$$

which differs from equation (32) only by a factor of 2 in the last term. Again this difference arises from the slightly different approximations we have used.

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INSTABILITY OF THE POSITIVE COLUMN IN A MAGNETIC FIELD AND THE 'ANOMALOUS' DIFFUSION EFFECT*

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Abstract—The positive column of a gas discharge is shown to become unstable when a sufficiently large longitudinal magnetic field is superimposed. This instability causes oscillations which lead to an increase in the flux of charged particles striking the walls; that is, in agreement with the experimental results, there is an increase in the effective diffusion.

1. INTRODUCTION

THE diffusion of electrons and ions across a magnetic field is a problem which is of considerable interest to plasma physics but it is one which up to the present time has yielded some very conflicting experimental results (GUTHRIE and WAKERLING, 1949; SIMON, 1955; BOSTICK and LEVIN, 1955; NEDOSPASOV, 1958; LEHNERT, 1959; HOH and LEHNERT, 1959; ZHARINOV, 1959; ELLIS *et al.*, 1959; SIRGII and GRANOVSKII, 1959 and VASIL'EVA and GRANOVSKII, 1959). In addition, in a number of experiments (GUTHRIE and WAKERLING, 1949; BOSTICK and LEVIN, 1955; LEHNERT, 1959; HOH and LEHNERT, 1959; ZHARINOV, 1959 and ELLIS *et al.*, 1959). an anomalously large diffusion of the plasma across the magnetic field has been observed. Consequently, experiments on the diffusion of the plasma in the positive column of a gas discharge are of special interest and in this connexion we would particularly cite the work of LEHNERT and HOH (1959) and also LEHNERT (1959). In the absence of an applied magnetic field this type of discharge is well understood both theoretically and experimentally. It therefore forms a convenient subject for theoretical treatment and for comparison with experiment. It is to this problem that the work reported here was devoted.

LEHNERT and HOH have measured the electric field E along the positive column as a function of the magnitude of the longitudinal magnetic field H. With magnetic fields which are not too strong they found that the electric field falls off as H increases in quantitative agreement with the theoretical form of the dependence of the ambipolar diffusion coefficient on the magnetic field strength. However, at a certain critical field H_c, which is of the order of several kilogauss, the character of the dependence of E on H was found to undergo a sudden change; the electric field increased with H and in some cases it actually exceeded the value at $H = 0$. This increase in E provides indirect evidence that the diffusion coefficient is considerably larger than the 'classical' value derived from a consideration of collisions between the electrons and ions with neutral gas molecules.

We show below that this effect can be explained by the instability of the positive column in the longitudinal magnetic field and by the development of oscillations of the 'diffusion' type.

2. INSTABILITY OF THE POSITIVE COLUMN

One can see from a simple qualitative argument that a longitudinal magnetic field can lead to a loss of stability of the plasma current channel. In the absence of a magnetic field any disturbance of the plasma will be rapidly terminated by the increased diffusion of particles from regions of excess density. In particular, the twisting and wriggling of the current channel causes an increase in the flux of particles striking the wall at those places where the channel comes close to it and a decrease in the flux on the reverse side. This has the effect of restoring the original state of the plasma. In the presence of a magnetic field there is an additional force $(1/c)\mathbf{j} \times \mathbf{H}$ acting on the plasma, where $\mathbf{j}$ is the current density. With a helical distortion of the plasma this force acts either towards the axis of the discharge or towards the walls according to whether the helix is right or left-handed. When this force acts in the direction of the walls there is a build-up of the initial disturbance. As the magnetic field is increased this force becomes greater and the transverse diffusion, and along with it, the stabilizing action of the walls is reduced. Thus, for sufficiently large magnetic fields the plasma column becomes unstable.

In order to investigate this stability problem we will make use of the diffusion equations for electrons and ions:

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v}_i = \frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v}_e = Zn. \qquad (1)$$

* A translation by D. L. ALLAN of this paper originally submitted in Russian.

Here n is the electron (and also the ion) density.

$Z = Z(T_e)$ is the number of ionization events per electron per unit time, T_e is the electron temperature, $\mathbf{v}_e$, $\mathbf{v}_i$ are the directional velocities of the electrons and ions which may be found from the equations of motion.

For the electrons, the equation of motion has the form:

$$\frac{kT_e}{mn}\nabla n = -\frac{e}{mc}\mathbf{v}_e \times \mathbf{H} + \frac{e}{m}\nabla V - \frac{\mathbf{v}_e}{\tau} \qquad (2)$$

where m is the electron mass, $1/\tau$ is the collision frequency and V is the electric field potential. With $\Omega\tau = (eH/mc)\tau \gg 1$, we obtain from equation (2):

$$v_{e\perp} = \frac{b_e}{\Omega\tau}\mathbf{h} \times \nabla V - \frac{D_e}{\Omega\tau}\frac{1}{n}\mathbf{h} \times \nabla n + \\ + \frac{b_e}{(\Omega\tau)^2}\nabla V - \frac{D_e}{(\Omega\tau)^2}\frac{\nabla n}{n}$$

$$v_{ez} = b_e\frac{\partial V}{\partial z} - \frac{D_e}{n}\frac{\partial n}{\partial z} \qquad (3)$$

where $\mathbf{h} = \mathbf{H}/H$ is the unit vector along the z-axis, b_e is the electron mobility and D_e is the electron diffusion coefficient.

We will further suppose that the frequency of the collisions between the ions and the neutral gas molecules $(1/\tau_i)$ is much greater than the cyclotron frequency eH/Mc and also much greater than the frequency of the oscillations we are considering. With this assumption we can neglect the effect of the magnetic field on the ions and also the inertia of the ions. We then obtain:

$$\mathbf{v}_i = -b_i\nabla V \qquad (4)$$

where b_i is the mobility of the ions. We have neglected here the diffusion of the ions. It can be shown that taking the diffusion of the ions into account introduces only a small correction of the order of the ratio of the ion temperature to the electron temperature.

Substituting the expressions for the velocities (3) and (4) into (1), we obtain two equations for n and V:

$$\frac{\partial n}{\partial t} + \frac{b_e}{\Omega\tau}\mathbf{h}\,.\,\nabla V \times \nabla n - \frac{D_e}{(\Omega\tau)^2}\Delta_\perp n + \frac{b_e}{(\Omega\tau)^2} \times \\ \times \operatorname{div} n\nabla_\perp V + \frac{\partial}{\partial z}\left(b_e n\frac{\partial V}{\partial z} - D_e\frac{\partial n}{\partial z}\right) = Zn \qquad (5)$$

$$\frac{\partial n}{\partial t} - b_i \operatorname{div} n\nabla V = Zn. \qquad (6)$$

In the equilibrium state n and V depend only on distance from the axis of the discharge tube. We will suppose that the mean-free-path of the ions λ_i is much less than the radius of the tube a. Then, as a boundary condition, we can equate the density at the wall to zero (Shottki condition), whereupon from (5) and (6) we obtain

$$n_0(r) = N_0 J_0(\beta_0 r), \qquad \frac{dV_0}{dr} = \frac{D_e}{b_e + b_i(\Omega\tau)^2}\frac{1}{n_0}\frac{dn_0}{dr}$$

$$Z = \frac{b_i}{b_e}\frac{D_e\beta_0^2}{1 + \frac{b_i}{b_e}(\Omega\tau)^2} \qquad (7)$$

Here $\beta_0 = (\alpha_0/a)$, α_0 is the first root of the zero order Bessel function J_0, N_0 is the density at $r = 0$.

In order to obtain the stability conditions we must find the frequency of the small oscillations of of the current channel. Because the equilibrium state has cylindrical symmetry we can choose a perturbation of the form $f(r)\exp(im\psi + ikz - i\omega t)$ where ψ is the azimuthal angle. The linearized equations have the form:

$$\left\{-i\omega - Z + ikv_0 + k^2 D_e + \frac{im}{r}\frac{b_e}{\Omega\tau}\frac{dV_0}{dr}\right\}n' - \\ - \frac{D_e}{(\Omega\tau)^2}\Delta_\perp n' + \frac{b_e}{(\Omega\tau)^2}\frac{1}{r}\frac{d}{dr}\left(rn'\frac{dV_0}{dr}\right) - \\ - \left\{k^2 b_e n_0 + \frac{im}{r}\frac{b_e}{\Omega\tau}\frac{dn_0}{dr} + \frac{m^2}{r^2}\frac{b_e n_0}{(\Omega\tau)^2}\right\}V' + \\ + \frac{b_e}{(\Omega\tau)^2}\frac{1}{r}\frac{d}{dr}\left(rn_0\frac{dV'}{dr}\right) = 0 \qquad (8)$$

$$(-i\omega - Z)n' + b_i k^2 n_0 V' + b_i\frac{m^2}{r^2}n_0 V' \\ - b_i\frac{1}{r}\frac{d}{dr}r\left(n_0\frac{dV'}{dr} + n'\frac{dV_0}{dr}\right) = 0. \qquad (9)$$

Here $v_0 = b_e E$ is the directional (current) velocity of the electrons in the equilibrium state and E is the longitudinal electric field.

To the equations (8) and (9) we must now add the boundary conditions. The first two boundary conditions imply that n' and V' are regular at $r = 0$, the third condition is that the density at the wall is zero and the fourth follows from the equality of the electron and ion currents to the wall. Since $(1/n)(\partial n/\partial r) \sim (1/\lambda_i)$ is a constant at the boundary, then the last condition is satisfied if the potential V' is finite at the wall.

An accurate treatment of the set of equations (8), (9) is rather difficult, and therefore we will find an approximate solution, specifying a certain radial dependence for n' and V'. We will confine ourselves here to

perturbations for which $|m| = 1$, since such perturbations will define the region of stability. It is also natural to choose a density dependence of the form $n' = n_1 J_1(\beta_1 r)$, where $\beta_1 = (\alpha_1/a)$ and α_1 is the first root of the Bessel function J_1. We will adopt the same form for V', that is $V' = V_1 J_1(\beta_1 r)$; the reason for this choice will be apparent later.

Let us substitute these expressions into equations (8) and (9), multiply them by $J_1(\beta_1 r) r\, dr$ and integrate with respect to r. We then obtain a set of two algebraic equations for n_1 and V_1:

$$\left\{-i\omega - Z + ikv_0 + k^2 D_e + \frac{D_e\beta_1^2}{(\Omega\tau)^2} - im\frac{L}{\Omega\tau} \times \right.$$

$$\left. \times \frac{D_e\beta_0^2}{1 + \dfrac{b_i}{b_e}(\Omega\tau)^2} - \frac{Q}{(\Omega\tau)^2}\frac{D_e\beta_0^2}{1 + \dfrac{b_i}{b_e}(\Omega\tau)^2}\right\}\frac{n_1}{n^0}$$

$$- \left\{k^2 b_e - imA\beta_0^2\frac{b_e}{\Omega\tau} + C\frac{b_e\beta_1^2}{(\Omega\tau)^2}\right\}V_1 = 0 \qquad (10)$$

$$\left\{-i\omega - Z + \frac{b_i}{b_e}\frac{QD_e\beta_0^2}{1 + \dfrac{b_i}{b_e}(\Omega\tau)^2}\right\}\frac{n_1}{n^0} + Cb_i\beta_1^2 V_1 = 0 \qquad (11)$$

Here n^0 is some average value of the unperturbed density and $A = 0{\cdot}66$, $C = 0{\cdot}79$, $L = 0{\cdot}74$ and $Q = 1{\cdot}56$ are the constants appearing after integrating.

In equation (11), we have neglected the motion of the ions along the z-axis in view of the fact that $b_i/b_e \ll 1$.

Putting the determinant of the set (10), (11) equal to zero we readily obtain the dispersion equation for ω. We have a stable condition only when the imaginary part of the frequency is negative. Using the dispersion equation, this condition can be written in the form:

$$KX^4 + FX^2 + G \geqslant mBv^* X\frac{b_e}{b_i} \qquad (12)$$

where

$$K = \frac{1{\cdot}28 + y}{y(y+1)}; \qquad F = \frac{0{\cdot}8\,(y+2)}{y};$$

$$G = \frac{0{\cdot}48(1+y)}{y} + 0{\cdot}1\,\frac{b_e}{b_i}\frac{1}{1+y} \qquad (13)$$

$$B = 0{\cdot}163; \quad X = \frac{k\Omega\tau}{\beta_1}; \quad v^* = \frac{v_0}{\beta_0 D_e}; \quad y = \frac{b_i}{b_e}(\Omega\tau)^2$$

From relation (12) it can be seen that the instability arises only when $m > 0$ and when the electron drift velocity v_0 is sufficiently large. The plasma is stable when there is no longitudinal current.

In a long tube arbitrary values X are possible. Therefore the plasma becomes unstable when the left-hand part of (12) turns out to be equal to the right-hand part at a unique point. In this case the derivatives of both parts will be equal too. Differentiating the equality (12) with respect to X, multiplying the result by X and subtracting one from the other we obtain a biquadratic equation for X, whence:

$$X^2 = \frac{-F + \sqrt{F^2 + 12KG}}{6K},$$

$$v^* = \frac{2b_i}{Bb_e}X(2KX^2 + F). \qquad (14)$$

Relation (14) determines the boundary of the stability region $v^* = v^*(y)$ where $v^* = (b_e/\beta_0)(E/D_e) = (eE/\beta_0 kT_e)$. The electron temperature is a function of E/p and is well-known experimentally (KLARFELD, 1941; KARELINA, 1941) and the same is true also of the mobilities b_i, b_e (ENGEL, 1955; GRANOVSKII, 1952). Using these experimental results we can calculate v^* and b_e/b_i for those values of E for which the monotonic dependence of E upon H has been observed to break down and then, using (14), we can find y and so find a theoretical value for the critical magnetic field H_e.

We carried out this calculation for helium. We took the values of the electric field corresponding to $H = H_e$ from the experimental results of HOH and LEHNERT and the values of T_e were taken from the work of KLARFELD. A comparison of the calculated values H_{et} with the experimental values H_{ee} is given in Table 1.

A result corresponding to the last row and first column of this table has not been computed, for here the conditions are such that it is not possible to ignore the action of the magnetic field upon the ions.

A similar calculation was carried out for argon but in this case there is only one experimental point: $H_{ee} = 3{\cdot}7$ kgauss with $a = 1$ cm and $p = 0{\cdot}92$ mm Hg. Once again the calculated value of the critical field $H_{et} = 3{\cdot}3$ kgauss agrees well with the experimental value.

Since all the coefficients in (12) are roughly of the same order of magnitude, X turns out to be of the order of unity, i.e. the instability develops for long wave perturbations $ka = (\alpha_0 X/\Omega\tau) \ll 1$. This explains why, in the experiments of LEHNERT and HOH, the dependence on the length of the tube L disappears only when $L/2a \geqslant 50$. For shorter tubes X is restricted to lower values. Thus, depending on the actual relations between the various parameters in (12) the plasma channel will either become stable again as H increases or it will be stable with respect to

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TABLE 1

ap	cm × mm Hg	0·24	0·45	0·89	1·45	1·6	1·8	3·0
a	cm	0·535	1	1	1	0·535	1	1
H_{cr}/p	kgauss/mm Hg	5·5	3·6	2·4	1·6	1·9	1·5	0·9
H_{cr}/p (He^+)	kgauss/mm Hg	5·1	2·8	2·0	1·6	1·5	1·4	1·0
H_{cr}/p (He^+_2)	kgauss/mm Hg	—	3·3	2·1	1·6	1·5	1·4	1·0

perturbations with $m = 1$ at all values of H. In the latter case, the loss of stability will occur when oscillations with $m > 1$ arise.

We will now return to the question of the choice of the radial dependence for V'. Let us multiply equation (9) by $[b_e]/[b_i(\Omega\tau)^2]$ add it to (8) and neglect small terms of the order

$$\frac{b_i}{b_e} y \sim \left(\frac{eH}{Mc}\tau_i\right)^2.$$

Then, by virtue of (7), we obtain:

$$n' \frac{D_e\beta_0^2}{b_e}\left\{\frac{\alpha_1}{\alpha_0} X v^* + \frac{m}{r(1+y)\beta_0^2}\frac{1}{n_0}\frac{dn_0}{dr}\right\} \approx \frac{m}{r}\frac{dn_0}{dr} V'. \tag{15}$$

By making use of the condition (12), one can show that when the plasma becomes unstable the left-hand side of (15) turns out to be everywhere positive except in a small region near the wall where it changes sign and goes to a finite negative value as $r \to a$. Since the density n' near the wall is small, this region makes only a small contribution while for the remaining values of r the function $J_1(\beta_1 r)$ is a satisfactory approximation for V'.

3 'ANOMALOUS' DIFFUSION

The oscillations of the plasma current channel which arise as a result of the instability lead to the appearance of an azimuthal electric field. The electrons are therefore able to drift in a radial direction which, when averaged, appears as an increased mobility across the magnetic field; in other words, we have an 'anomalous' diffusion.

For the purpose of providing a quantitative description of this effect we can no longer use the linearized diffusion equations. Provided that the magnetic field is not much larger than H_c, the amplitude of the oscillations will be small and can be treated as harmonic. Then, as before, we can write

$$n = n_0(r) + n_1 \cos(\psi + kz - \omega t) J_1(\beta_1 r),$$
$$V = V_0(r) + V_1 \cos(\psi + kz - \omega t + \delta) J_1(\beta_1 r) \tag{16}$$

where the phase shift δ can be considered to be approximately independent of r.

Substituting these expressions into equations (5) and (6) and averaging over time, we obtain

$$\frac{1}{r}\frac{d}{dr} r\left\{\frac{b_e}{(\Omega\tau)^2} n_0 \frac{dV_0}{dr} - \frac{D_e}{(\Omega\tau)^2}\frac{dn_0}{dr} + \frac{b_e}{\Omega\tau}\frac{n_1 V_1}{r}\frac{\sin\delta}{2} \times \right.$$
$$\left. \times J_1^2(\beta_1 r) + \frac{b_e}{(\Omega\tau)^2}\beta_1 n_1 V' J_1(\beta_1 r) \times \right.$$
$$\left. \times J_1'(\beta_1 r)\frac{\cos\delta}{2}\right\} = Z n_0 \tag{17}$$

$$-\frac{1}{r}\frac{d}{dr} r\left\{b_i n_0 \frac{dV_0}{dr} + b_i n_1 V_1 \beta_1 J_1(\beta r) \times \right.$$
$$\left. \times J_1'(\beta r)\frac{\cos\delta}{2}\right\} = Z n_0. \tag{18}$$

When the amplitude of the oscillations is small, the approximate solutions of equations (17) and (18) can be found by putting

$$n_0 = N_0 J_0(\beta_0 r), \quad \frac{dV_0}{dr} = \frac{1}{n_0}\frac{dn_0}{dr}\frac{D_e}{b_e(1+y)} S,$$

where $S = \text{const}$.

We will now insert this into equations (17) and (18) multiply them by $J_0(\beta_0 r) r\, dr$ and integrate with respect to r. From the equations obtained we can find Z and S:

$$Z = \frac{b_i}{b_e}\frac{D_e\beta_0^2}{1+y}\phi, \quad S = \phi - \rho\frac{(\phi-1)(1+y)}{\Omega\tau}\operatorname{ctg}\delta. \tag{19}$$

Here $\phi = 1 + \Omega\tau\, s\,.\sin\delta$, s is a small quantity of the order of the amplitude of the oscillations and $\rho = 2{\cdot}6$ is a constant which is expressed in terms of the integrals of the Bessel functions.

In the equations for small oscillations we can neglect the non-linear terms since they contain second harmonics which disappear when averaged over a half-cycle. Thus n' and V' may once more be considered as complex quantities and the only change we

have to make to equations (8) and (9) is to replace Z and dV_0/dr by the new values.

Making this change and repeating the calculation, we obtain the following dispersion equation ($m = 1$):

$$\frac{i\omega}{C}\left\{\frac{C(1+y)}{y}+\frac{X^2}{y}-iA\left(\frac{\alpha_0}{\alpha_1}\right)^2\frac{b_e}{b_1\Omega\tau}\right\}=\frac{D_e\beta_1^2}{(\Omega\tau)^2}\times$$

$$\times\left\{i\Omega\tau\frac{\alpha_0}{\alpha_1}v^*X+1-\left(\frac{\alpha_0}{\alpha_1}\right)^2\phi+X^2\left[1+\left(\frac{\alpha_0}{\alpha_1}\right)^2\times\right.\right.$$

$$\left.\left.\times\frac{SQ-\phi}{C(1+y)}\right]-\frac{i\Omega\tau}{1+y}\left[SL+A\left(\frac{\alpha_0}{\alpha_1}\right)^2\frac{SQ-\phi}{C}\right]\right\} \tag{20}$$

The condition Im $\omega = 0$ takes the form:

$$\frac{1+y+0{\cdot}78S-0{\cdot}5\phi}{y(1+y)}\cdot X^4+$$

$$+\frac{0{\cdot}8y+1{\cdot}8+0{\cdot}6S-0{\cdot}8\phi}{y}\cdot X^2+0{\cdot}8(1-0{\cdot}4\phi)\times$$

$$\times\frac{1+y}{y}+0{\cdot}1\frac{b_e}{b_i}\frac{1{\cdot}3S-0{\cdot}3\phi}{1+y}=0{\cdot}163\frac{b_e}{b_i}Xv^* \tag{21}$$

It then follows that the frequency of the oscillations given by equation (20) is approximately

$$\omega\approx\frac{b_i}{b_e}\frac{3D_e\beta_1^2}{\Omega\tau}\{1-0{\cdot}4\phi+X^2\} \tag{22}$$

From the equation for the ions it is not difficult to find the phase shift δ. It is given by the relation:

$$\operatorname{ctg}\delta=-\frac{b_i}{b_e}\frac{D_e\beta_0^2}{(1+y)\omega}(QS-\phi). \tag{23}$$

Inserting (22) and (23) into (19) we find the relation between S and ϕ:

$$S=\phi\frac{1{\cdot}3+X^2-0{\cdot}7\phi}{1{\cdot}5+X^2-0{\cdot}9\phi} \tag{24}$$

By specifying a definite value for ϕ and substituting (24) into (21) we obtain an equation for X and y. Stationary oscillations are possible only where equation (21) is satisfied at the unique point where the curves defined by the right and left-hand sides of (21) touch tangentially. From this condition we can determine the theoretical curve $v^* = v^*(y, \phi)$. On the other hand we know the form of the dependence $v^*(E/p)$ experimentally and, furthermore, according to (19)

$$E/p=f\left(\sqrt{\frac{1+y}{\phi}}\,ap\right),$$

where $f(ap)$ is the experimentally determined dependence of E/p on ap in the absence of a magnetic field.

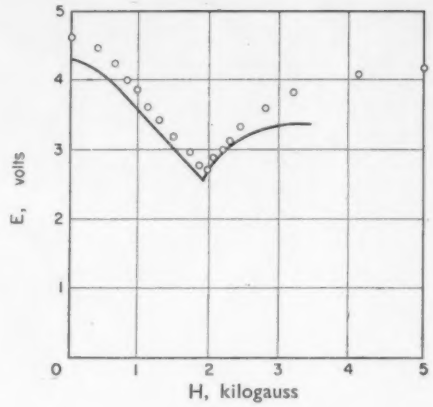

FIG. 1.

We will find $y = y(\phi)$, and consequently the relation between E and H in the presence of steady oscillations, by finding the intersection of the curve $v^* = v^*(y)$ calculated in this way with the curve defined by the condition Im $\omega = 0$.

In Fig. 1 the curve obtained on the basis of these calculations is compared with the experimental results of HOH and LEHNERT for the values $a = 1$ and $p = 0{\cdot}89$. It can be seen that there is not only qualitative agreement with the experimental points but, furthermore, good quantitative agreement up to a field $H = 3$ kgauss. At this value of the field $\phi = 4$, i.e. the averaged flux of particles at the wall is four times greater than the diffused flux. The departure of the theoretical curve from the experimental points when H is increased further may be due either to the approximations made in solving the equations or to the appearance of oscillations having a shorter wavelength (in particular, those with $m > 1$).

4. SCALE RELATIONS

Equations (5) and (6) describing the diffusion processes in the plasma have the specific property that length, time and magnetic field enter into them only in combination with the pressure; that is, in the form ap, tp, H/p. This is because Z is proportional to, and b, D and τ inversely proportional to the pressure. Equations (5) and (6) still depend implicitly on the electron and ion temperatures and these are determined by the energy balance, i.e. ultimately by the quantity E/p.

This means that if we have two geometrically similar systems with identical parameters ap, E/p and H/p equations (5) and (6) coincide for both systems and the processes which take place in these systems will differ only by the time scale. This is still the case when the action of the magnetic field on the ions and the diffusion and inertia of the ions are taken into account.

Furthermore, equations (5) and (6) are linear with

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respect to the density n, i.e. the absolute magnitude of n or, in other words, the magnitude of the electric current in no way affects the processes which take place in the column.

Thus, if the degree of ionization is so small that electron-ion collisions can be neglected, then we have the following scale relation: E/p and ω/p are functions of ap and H/p only and are independent of the magnitude of the current. A comparison of the experimental dependence of E on H obtained by HOH and LEHNERT for different values of a and p shows that such a scale relation is in fact valid. This can also be seen from the table of critical field values H_c (Table 1).

5. CONCLUSION

We have seen that the positive column of a gas discharge becomes unstable if the magnetic field exceeds a certain critical value H_c. If the magnetic field is only a little greater than H_c then regular oscillations build up in the plasma with a frequency

$$f = \frac{\omega}{2\pi} \sim \frac{10 b_i D_e}{\Omega \tau b_e a^2}$$

which, for helium and the experimental conditions of LEHNERT and HOH, is roughly equal to 10 kc/sec. It seems possible that such oscillations would explain the effect observed by. FABRIKANT and ROKHLIN (1938) who found that the maximum electron density occurred off the axis of the discharge when a magnetic field was applied.

One might suppose that with a further increase of H even higher harmonics will be excited and the oscillations of the plasma channel will eventually become irregular and chaotic. Such a situation can be identified with the turbulent state of the plasma first mentioned by BOHM (GUTHRIE and WAKERLING, 1949). By analogy with normal turbulence, one might attempt to formulate a theory of such a turbulent discharge but to do this it would be desirable to have more complete experimental data on the intensity and spectrum of the oscillations.

Acknowledgement—This work was carried out after discussing the various problems mentioned in this paper with V. D. SHAFRANOV. To him the authors extend their very sincere thanks.

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THE INJECTION OF PLASMOIDS INTO A MAGNETIC TRAP WITH A FIELD WHICH INCREASES TOWARDS THE PERIPHERY*

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Abstract—The feasibility of filling up a magnetic trap with plasma has been investigated by injecting accelerated plasmoids into a trap formed by a magnetic field which increases towards the periphery. It is shown that in such a system the lifetime of the plasma is several tens of microseconds under the conditions of these experiments.

1. INTRODUCTION

THE behaviour of a plasma within a trap formed by two solenoids connected in opposition has already been described in an earlier paper (LUKYANOV and PODGORNYI, 1959). In the work reported here we examined the possibility of filling such a trap with plasma by injecting plasmoids at a velocity sufficient for them to pass through the magnetic barrier. A plasmoid which has penetrated into a zero field region may then find itself confined within the trap as a result of the occurrence of irreversible processes.

The experiments to be described here were carried out with the aim of studying some processes by which the plasma is confined within a trap formed by a magnetic field which increases towards the periphery. In addition, we wished to find out more about the physical processes associated with the penetration of plasmoids through the magnetic barrier into the low field region.

2. APPARATUS

The vacuum chamber of the 'accelerated plasmoids trap' (LUS–1) was made of stainless steel and had a height of 100 cm and a diameter of 21 cm. The magnetic field was created by two solenoids which, in order to obtain the required field shape, were connected in opposition. At a current of 400 amp through the windings of the solenoids the maximum magnetic field in the space between them was 1500 oersted. During the experiments the pressure of the residual gas within the vacuum chamber was less than $1–2 \times 10^{-6}$ mm Hg.

The plasmoids were injected into the trap using a coaxial electrodynamic accelerator, details of which are to be published shortly (the work on this machine was carried out in conjunction with S. A. Chuvatin). The electrodes of the injector were connected directly (i.e. no discharge switch was used) to a condenser bank with a working voltage up to 12 kV. For most of the experiments to be described here the capacity of the condenser bank was 2μF. A measured quantity of hydrogen was passed through a high-speed mechanical valve which formed an integral part of the plasma injector. Under normal conditions the amount of gas introduced into the injector was $\sim$ 0·1 cm³. The discharge current in the injector circuit was measured by a Rogovsky belt.

The radiation emitted by the plasma in the visible region of the spectrum was recorded by photomultiplier tubes which were so collimated that only quanta originating in a definite region of the magnetic trap could reach their photocathodes. The position of these photomultipliers is shown in Fig. 1. In order

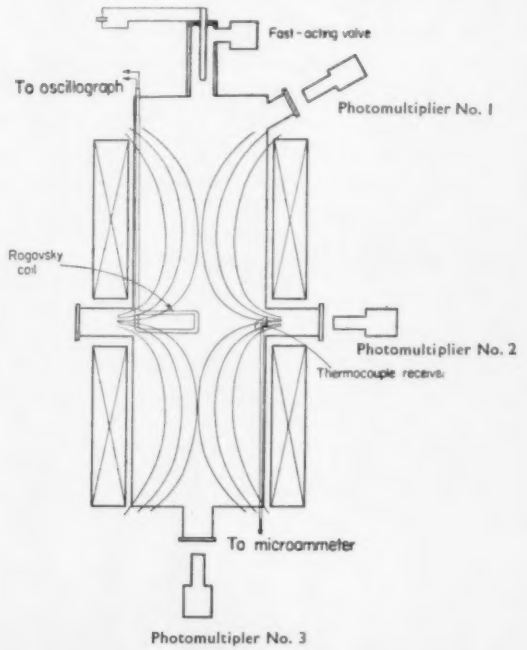

FIG. 1.—Apparatus.

*A translation by D. L. ALLAN of this paper, originally submitted in Russian.

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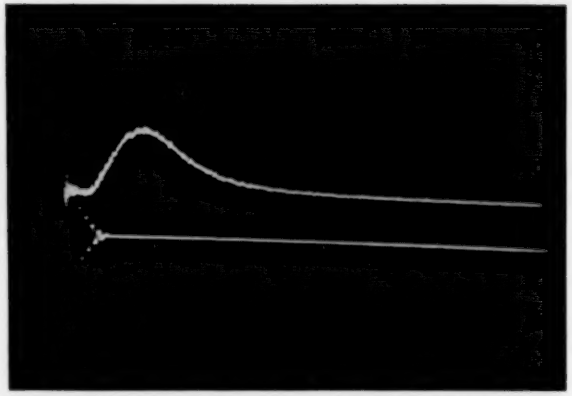

FIG. 2.—Oscillogram of impulse from photomultiplier No. 2 and of the discharge current in the injector ($C_0 = 2{\cdot}5\,\mu$F, sweep time $\sim$250 μsec).

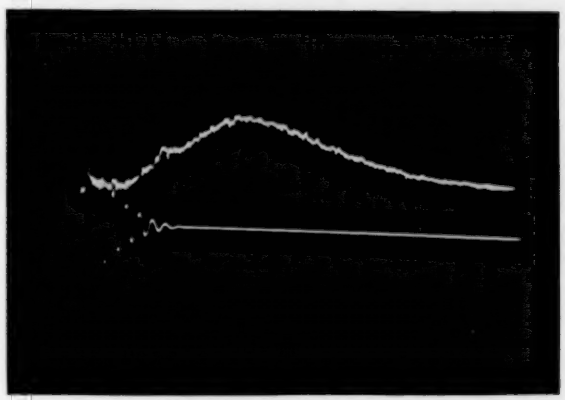

FIG. 3.—Oscillogram of impulse from photomultiplier No. 2 and of discharge current in the injector ($C_0 = 2{\cdot}5\,\mu$F, sweep time $\sim$120 μsec).

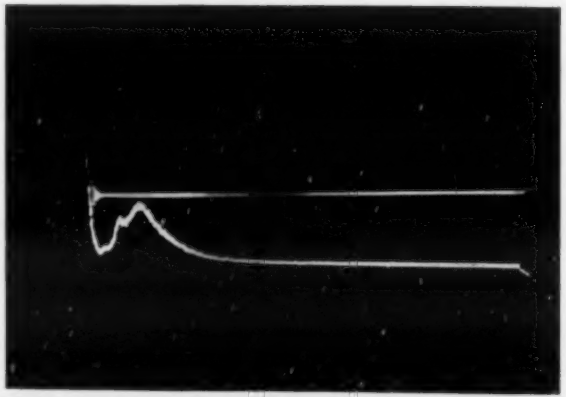

FIG. 4.—Oscillogram of discharge current in the injector and of light pulse from photomultiplier No. 3 ($C_0 = 0{\cdot}6\,\mu$F; sweep time $\sim$250 μsec).

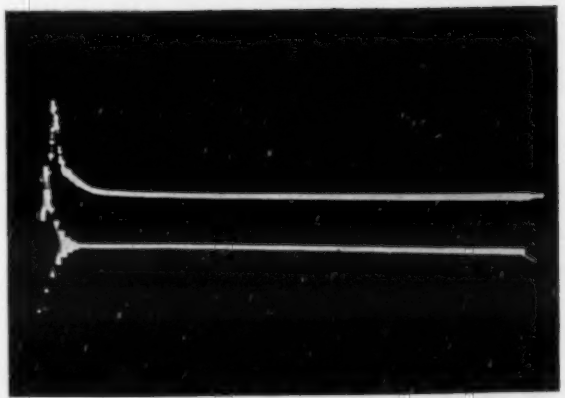

FIG. 5.—Oscillogram of light impulse from photomultiplier No. 1 and of discharge current in the injector ($C_0 = 2\,5\,\mu$F, sweep time $\sim$350 μsec).

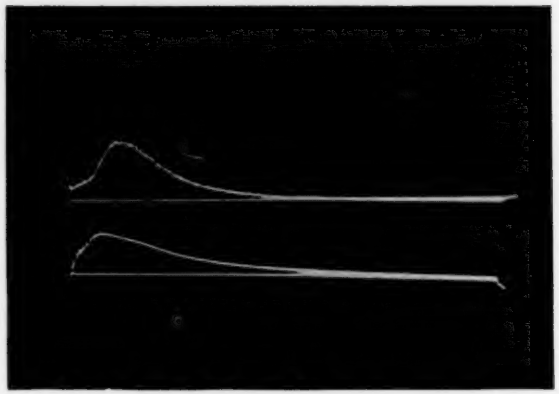

FIG. 6.—Oscillogram of impulse from photomultiplier No. 2 and of the saturation current to the probe at a probe potential with respect to the walls of the chamber equal to 35 V ($C_0 = 0{\cdot}6\ \mu$F, sweep time $\sim 250\ \mu$sec).

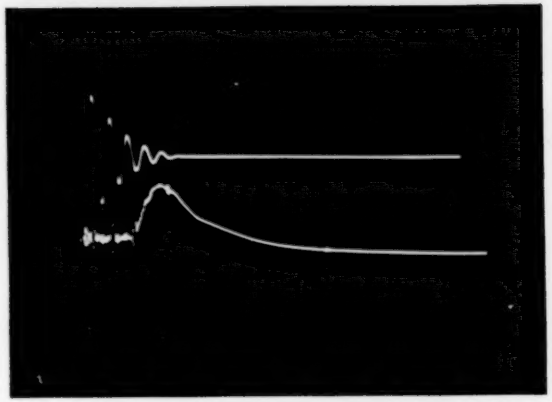

FIG. 7.—Oscillogram of discharge current in the injector and of the pulse of current in the plasma ($C_0 = 2{\cdot}5\ \mu$F, sweep time $\sim 120\ \mu$sec).

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to record radiation emitted in a narrow spectral interval, a type UM–2 monochromator was interposed between the collimator and the photomultiplier tube. Photographs of the spectra in the visible region were taken using a type ISP–51 spectrograph.

To measure the flux of heat reaching the wall of the vacuum chamber we used a thermocouple, the heat receiver of which could be moved near the wall inside the vacuum chamber. The height of the plate forming the heat receiver was 5 mm.

The currents flowing in the plasma were measured by small Rogovsky belts which could be inserted in the vacuum chamber when required. Naturally, the introduction of foreign bodies into the volume occupied by the plasma affected its behaviour and consequently the results obtained using Rogovsky belts should be treated with a certain amount of reserve. In order to study the localized currents in the plasma we used a Rogovsky belt which could be moved along the axis of the chamber. Both the belt and the thermocouple could be moved about inside the chamber without breaking the vacuum.

The behaviour of the plasma as a function of time was studied using an OK–17M pulse oscillograph. Coaxial cable was used to connect up the measuring instruments and all leads and connexions were carefully shielded.

3. RESULTS

We began our study of the plasmoid injection process by investigating the time dependence of the intensity of radiation emitted in the visible region of the spectrum. With the field producing solenoids energized, a light impulse from the photomultiplier No. 2 appeared several microseconds after current had begun to pass in the injector circuit (Figs. 2 and 3). This delay before the appearance of the light pulse is caused by the flight time of the plasmoid. At first the intensity of the radiation increases rather rapidly, it reaches a maximum and then dies away exponentially with a damping decrement of about 20 μsec. In those cases where the period of the current oscillations was short (injector circuit condenser capacity 0·6 μF) the maximum radiation intensity occurred after the current in the injector had ceased.

With one or both solenoids switched off (in other words, in the absence of a trapping field) but with exactly the same amplifier gain in the light recording circuit, photomultiplier No. 2 failed to record any radiation from the plasma. In the absence of a magnetic field, it was necessary to increase the amplifier gain several hundred times before light pulses were recorded from the travelling plasmoids.

With the solenoids switched on, an examination of the spectral composition of the emitted radiation showed strong H_α and H_β hydrogen lines against a weak background. In addition, the ionized carbon lines at 4267 Å and 6582 Å were clearly seen. It was found that the time dependence of the individual spectral line intensities was the same as that of the continuous background and therefore, in later work, all measurements on the time dependence of the radiation intensity were carried out without placing a monochromator in front of the photomultiplier. A study of the variation of the plasma luminosity with magnetic field strength showed that for magnetic fields between 100 and 1500 oersted the damping decrement remained almost constant at the value already mentioned (about 20 μsec).

Rather different results were obtained from photomultiplier No. 3, which was placed below the chamber in a position where it recorded not only the light from the space inside the vacuum chamber but also the radiation emitted during the discharge in the injector (see Fig. 4). At the instant the gas in the injector breaks down there is an intense burst of radiation which dies away at the same rate as the current. After this, the output signal of the photomultiplier rises again. This fact as well as the increase in the intensity of the radiation recorded by multiplier No. 2 would appear to show that there is an accumulation of plasma inside the trap. One expects to find that the passage of the plasmoids through the magnetic barrier is accompanied by an increase in the light impulse recorded by tube No. 1 compared with the light impulse recorded by this photomultiplier when the magnetic field created by the upper coil is removed. An increase in the luminosity of the light burst may occur not only as a result of an increase in the density of the plasma but also because of the reduction in the velocity of the plasmoid as it penetrates into the interior of the magnetic field. Such an increase in luminosity was actually observed. Fig. 5 shows an oscillogram of the pulse recorded by tube No. 1 with the magnetic field switched on and it will be seen that no light was recorded by this photomultiplier after the injector current had died away. This is not surprising because it is not possible for any of the radiation emitted in the zero field region to reach the photocathode of this photomultiplier.

After these preliminary experiments had demonstrated that the plasma could be captured and contained within a trap, we were in a position to begin a more systematic study. We inserted a Langmuir probe near the point of zero magnetic field and obtained probe characteristics by measuring the current to the probe as a function of the potential difference

applied between the probe and the walls. Then, using the usual methods, we deduced the electron temperature and the charged particle density from the probe characteristics.

These quantities were also determined by the double probe method, (JOHNSON and MALTER, 1950), the results of the two methods agreeing to within the limits of the experimental errors. The maximum value found for the electron concentration was 3×10^{12} cm^{-3} at an electron temperature $\sim$10 eV. These values were obtained with a voltage $V_0 = 8{\cdot}5$ kV applied to the injector condenser bank ($C = 2\ \mu$F) and a magnetic field in the slit, $B = 1000$ oersted. It should be noted that the probe current time-dependence curve is similar to the time-dependence curve for the intensity of the radiation emitted in the visible region of the spectrum. The damping decrement of the probe current was found to be roughly 40 μsec (Fig. 6). Both the plasma luminosity and probe measurements clearly demonstrate the effect of plasma capture and confinement within the trap. The next stage in these experiments was to measure the heat flux incident upon the wall of the chamber. This was done by moving the heat receiver of the thermocouple near the wall in the direction parallel to the axis of the chamber.

The thermocouple was moved to different positions and, by measuring the heat received by the thermocouple at various places, a curve giving the distribution of heat flux to the wall was obtained. In some preliminary measurements we had shown that no current flowed to the walls during the discharge of the condenser bank in the injector circuit. Such a flow of current could have caused considerable heating of the thermocouple receiver and would have upset the heat flux measurements.

These experiments showed that there was a sharp peak in the heat flux distribution curve at a position corresponding to the gap between the solenoids. The width of this peak is determined partly by the width of the magnetic slit through which the outgoing charged particles pass and partly by the dimensions of the heat receiver of the thermocouple which, in these experiments, was 5 mm in height. The theoretical expression for the width of the magnetic slit (FIRSOV, 1958) is $\delta = 4\rho\sqrt{\left(1 + \frac{\pi Dt}{4\rho^2}\right)}$ where $\rho = \frac{mvc}{eB}$ is the Larmor radius for electrons and D is the coefficient of diffusion in a direction perpendicular to the magnetic lines of force. Fig. 8 presents the results of the experimentally determined half-widths of the heat flux distribution peaks measured at the wall for various magnetic field strengths. The experimental points can be well fitted by the straight line $\delta = \delta_\infty + \mathrm{K}(I/B)$. Extrapolating this line to infinitely large values of B gives $\delta_\infty = 5$ mm which, as one would expect, corresponds to the dimensions of the thermocouple heat receiver.

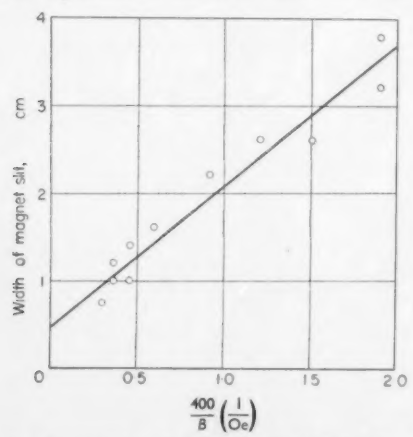

FIG. 8.—Dependence of magnetic slit width on the magnetic field strength.

The experimentally determined dependence has the same form as that given by the theory (FIRSOV, 1958) provided it is assumed that the slit expands as a result of classical diffusion and if one takes into account the fact that the lifetime of the plasma does not depend on the magnetic field strength. Here we suppose, in accordance with experimental data, that the ratio $\frac{n}{\sqrt{T_e}}$ which determines the diffusion coefficient

$$D = \frac{4 \cdot 10^{-4} n}{\sqrt{T}\, B^2},$$

is insensitive to the magnetic field strength in the region investigated. The fact that the damping decrement remains constant as the magnetic field is varied would seem to indicate that in this case the time the plasma persists is determined not by the escape of particles through the magnetic slit but by other processes such as charge exchanging, radiation etc. If we substitute the experimental values of the plasma lifetime, electron temperature and electron concentration, and use the expression for the classical diffusion coefficient (GUTHRIE and WAKERLING, 1949) we find that we obtain good agreement not only with the general form of the theoretical dependence $\delta = f(B)$ but obtain good absolute agreement as well. For example, the calculated slit width for $B = 1000$ oersted is 4 mm while the experimental value obtained after subtracting the height of the heat receiver is 6 mm.

It will be remembered that in the earlier work, (LUKYANOV and PODGORNYI, 1959), in which a beam

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of electrons was injected into the trap, a different type of dependence of the magnetic slit width on the magnetic field strength was obtained. It was found that for magnetic fields above 500 oersteds and for electrons with energies up to 200 eV, the width of the slit was independent of the magnetic field. This is due to an electron-optical image of the cathode surface in the side wall of the chamber.

A flow of current must occur in the plasma when the plasmoids pass through the magnetic field region and become confined within the traps. Experiments showed that a closed current began to flow in the plasma as soon as the plasma glow appeared (Fig. 7) and that this ring current was localized in the space between the magnetic field solenoids. This current died away with the time constant $\tau \sim 20$ μsec. The direction of the current in the plasma was the same as that in the upper solenoid. The direction did not depend on the polarity of the injector circuit condenser bank. The amplitude of the current fell by 20 to 30 per cent when the magnetic field was increased from 100 to 1500 oersteds and the peak value was around 300 A ($V_0 = 12$ kV). When one or both solenoids were switched off, the current disappeared and no current was detected when the condenser bank voltage was less than 6 kV. For voltages higher than this, a current pulse was always observed, the peak value rising monotonically increasing with voltage.

There may be several factors which cause the ring current in the plasma to die away. If Joule losses are the main cause of the current decay, we can find the conductivity of the plasma in the region where the current is flowing from the known geometrical size of this region as measured by a small Rogovsky belt. By substituting this experimental value of the conductivity into the formula expressing the conductivity in terms of the electron temperature, we deduced that the electron temperature was $T_e \sim 15$ eV. To within the limits of the experimental errors this value agrees with the probe measurement results and also with the results for the energy of the ions obtained by measuring the velocity of the plasmoids. It was assumed that a considerable fraction of the kinetic energy of the plasmoid is transferred to the electrons as thermal energy.

The results obtained in this work demonstrate the existence of a plasma confining effect in a magnetic trap formed by a magnetic field which increases towards the periphery. Some of the results which have been presented do not appear to fit in with the existing ideas on the subject. For example, although the fact that a current pulse arises within the plasma can be explained by the penetration of the plasmoid through the barrier, the reason why the current pulse lasts for some time is not so clear. It is obvious that the whole problem of the decay and hence the lifetime of a trapped plasma of the type discussed here requires further careful investigation.

Acknowledgements—The authors thank L. A. Artsimovich, S. Iu. Lukyanov and O. B. Firsov for interesting discussion.

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LETTERS TO THE EDITOR

The confinement of shock-heated plasmas in mirror magnetic fields

(*Received* 25 *February* 1960)

A POSSIBLE method of producing a hot plasma is to heat a suitable gas such as deuterium by means of the shock wave driven by the rapidly rising magnetic field of an electrodeless gas discharge. Plasma leakage is reduced by shaping the surrounding conductor so that the magnetic field is greatest at the ends. A well-known experimental example is the non-uniform θ-pinch Scylla (ELMORE *et al.*, 1958). Although these devices resemble the so-called 'mirror machines' in the general method of plasma containment, a detailed study of plasma loss reveals possible major differences of behaviour.

We shall consider here the case in which the central plasma density and temperature are such that the ion scattering time is so small that the ion pressure is isotropic. We can then consider the plasma as a fluid and work out the consequential behaviour.

The ions are held in largely by space-charge forces which act across the current sheath. Such forces cannot of course act along the lines of force and the plasma can escape laterally within the current sheath. The effective area of the hole at the end of the device can be shown to be approximately

$$A_E = A_M <\xi>/R, \tag{1}$$

where $A_M = \pi r_t^2$ is the area of the middle of the flux tube which bounds the plasma, R is the mirror ratio (defined as the ratio of fields when there is no plasma present) and $<\xi>$ is a dimensionless flux average defined by the relation

$$<\xi> = \int_0^{r_t} B_p \, 2\pi r \, dr / \pi r_t^2 \, B_M, \tag{2}$$

where B_p is the field in the plasma and B_M the field outside the plasma at the middle of the device. The integral is evaluated at the middle. Equation (1) follows quite simply by considering the plasma to be bound to the lines of force and applying Bernouilli's theorem to the plasma which flows out of the device.

Ions and electrons escape at the same speed i.e. the speed of the ions because the ions have the mass.

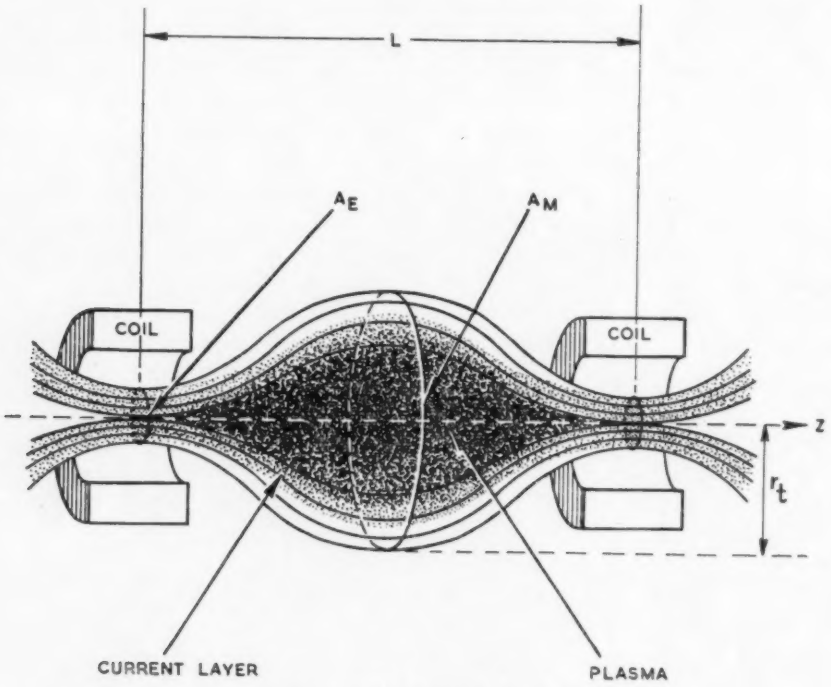
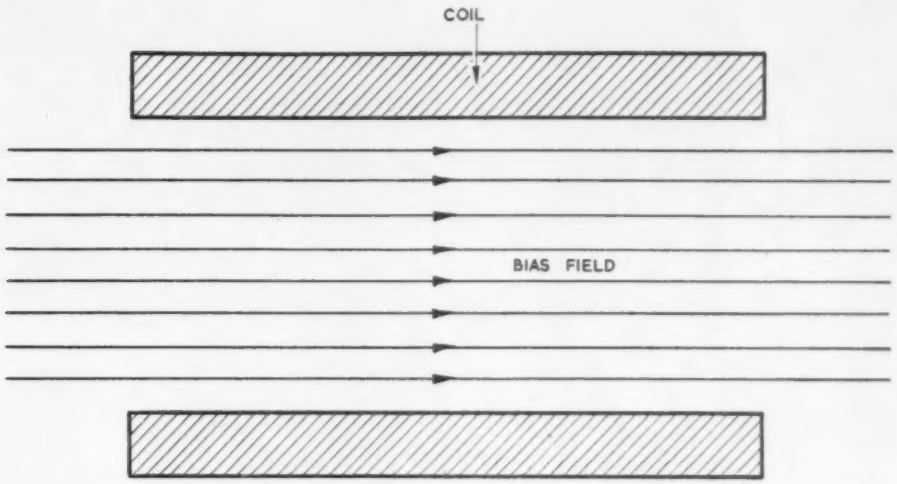


FIG. 1.—Section of non-uniform theta pinch.

The customary concept of a mirror machine is a device in which the plasma is well mixed with its confining magnetic field and the ratio of plasma pressure to the pressure of magnetic field in the plasma β, is small compared to unity. Particles whose velocity vectors lie in the so-called 'loss cone' can escape from the system. Particle losses are essentially determined by the scattering time for ions into the loss cone and hence in low-density, high-temperature plasma, long containment times are possible.

In the shock-heated device, the magnetic field may be largely external to the plasma. We then have a central region of high density plasma surrounded by a region of mixed field and plasma (see Fig. 1).



(a)

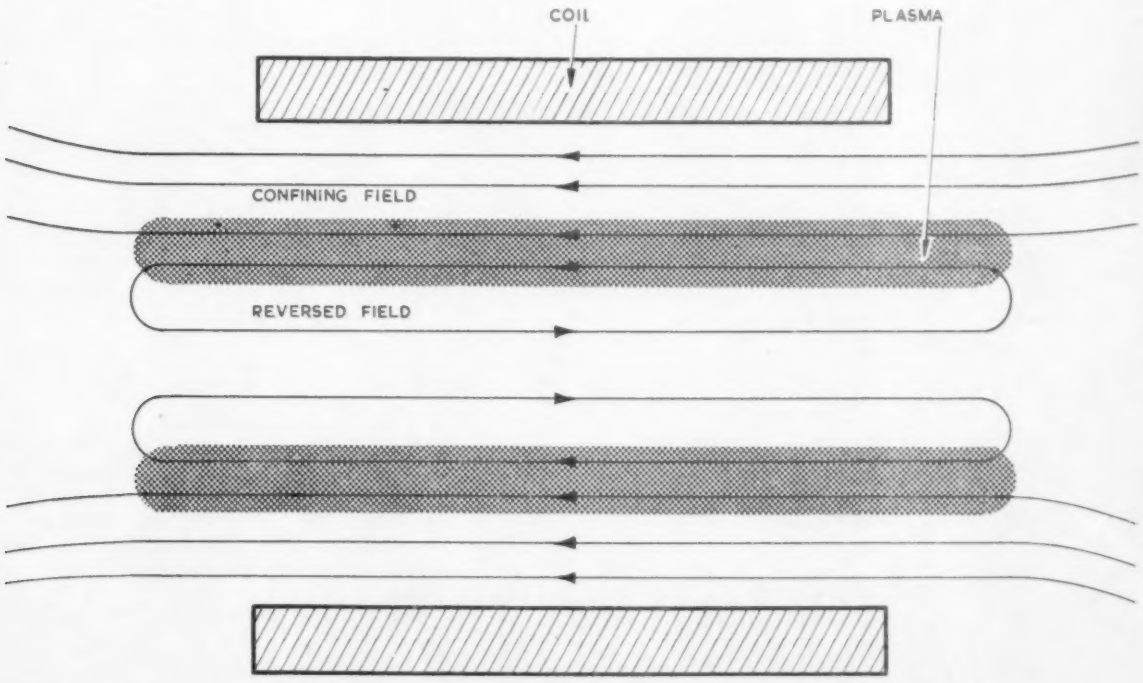
FIG. 2(a).

If particles escape axially with a speed $v_{||}$, the containment time of the plasma is roughly

$$\tau_e = LA_M/2\,A_E v_{||} = LR/2v_{||}<\xi>, \qquad (3)$$

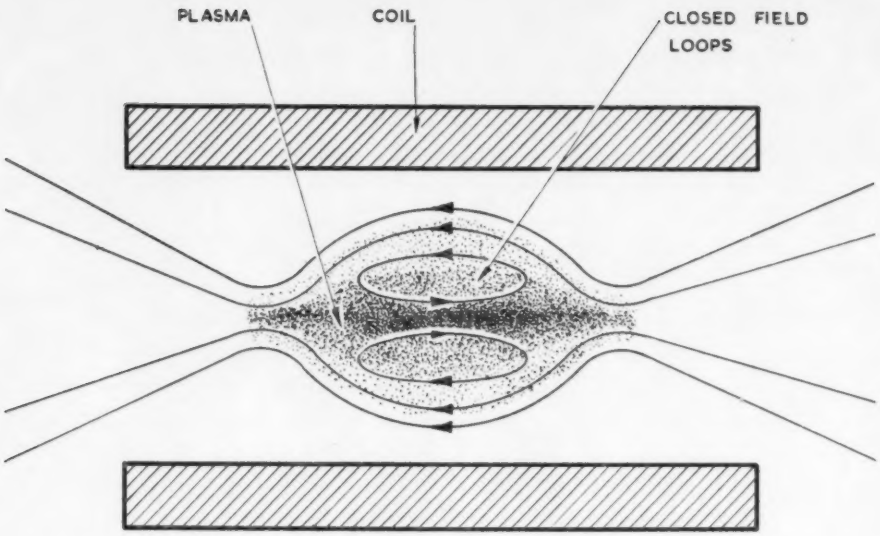
a result which apart from the factor $<\xi>$ was obtained by KOLB (1959). It follows at once from the result given in equation (3), that the exclusion of flux from the plasma improves plasma containment.

On this simple picture, we also see that closed loops of field in the plasma due to the trapping of magnetic field of reversed



(b)

FIG. 2(b).



(c)

FIG. 2(c).

sense to the containing field have the effect of lowering $<\xi>$ and thus enhance plasma containment. The sequence of events which gives rise to this situation is shown in Fig. 2 (a,b,c).

Before the pinch collapses, a reversed bias field may be applied to the discharge tube (a). This field may be provided by separate coils or by the residual field from a previous half-cycle of the discharge. The current from the main condenser bank then flows through the surrounding coil and the main magnetic field compresses the gas after breakdown. Lines of force of the reversed field inside the discharge link up with lines of the compressing field which have penetrated the plasma (b). The compression continues and the closed field loops shrink axially because of diffusion. Finally in (c), we see the plasma compressed with the closed loops inside it. In three dimensions this internal configuration is just the same as a toroidal Z-pinch. These loops will rapidly disappear since firstly, the rate of dissipation inside them is much enhanced because of field reversal (this is because the magnetic field has a zero point or points inside the loop and hence the gradient of field is always maintained at a high level) and, secondly, the whole internal configuration is likely to be unstable, probably resulting in turbulent dissipation. Before the loops have disappeared, the plasma inside them is effectively contained because it has to cross the lines of force to escape. Of course, once they have disappeared following dissipation, the containment is still improved by comparison with the case in which non-reversed trapped field exists because the plasma is left effectively field-free.

We now treat the case which apparently exists in Scylla i.e. a plasma surrounded by a thin skin of field and plasma (perhaps due to reversed field trapping described above).

We evaluate the penetration depth of the confining field by observing that the diffusion of field into the plasma is governed by the equation

$$(m_e c^2/4\pi n_e e^2)\frac{\partial}{\partial t}(\nabla^2 B) + (\eta/4\pi)\nabla^2 B = \frac{\partial B}{\partial t} \qquad (4)$$

where n_e is the electron density in the plasma, m_e the electron mass and η the plasma resistivity. In the above equation we have taken electron inertia into account (alternatively, we have not neglected the finite electron larmor radius).

We can roughly represent the solution of this equation by saying that if the field starts off at time $t = 0$ completely external to the plasma, then the penetration depth at time t is approximately

$$\lambda = (\eta t/4\pi + \lambda_p^2)^{1/2}, \qquad (5)$$

where

$$\lambda_p = (m_e c^2/4\pi n e^2)^{1/2}$$

is the plasma wavelength.

The first term in the bracket on the R.H.S. of equation (5) is the square of the ordinary resistive skin depth. We see that λ cannot be less than λ_p. This is obvious when we remember that under perfectly conducting conditions, λ_p is the mean electron Larmor radius.

The resistive part of the skin depth will increase until the axial flux of particles out of the sheath balances the radial diffusion of particles into the sheath i.e.

$$4\pi r_i \lambda n v_{\parallel}/R = 2\pi r_i L n(d\lambda/dt) \qquad (6)$$

To see this we must appreciate that if particles flow out of the sheath without being replaced, the requirements of pressure balance ensure that the gradient of magnetic field increases and there is enhanced dissipation. Conversely if not enough particles flow out of the sheath then the sheath thickens and the diffusion of the surrounding field slows down.

Hence the resistive part of the skin depth increases until the time t is given by

$$t = RL/2v_{\parallel} = R\tau_{ii}$$

where τ_{ii} is an ion transit time from the middle of the device to the end. At this point, we must question our original concept of the fluid being bound to the lines of force. However equation (6) implies that the diffusion time of an ion across the sheath is

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always greater than or equal to the ion transit time, thus our approximation is not a bad one.

The area of the hole follows from equation (1) and is

$$A_E = 2\pi r_t \lambda / R, \qquad (\lambda \ll r_t).$$

If we remember that the resistivity can be written in the form

$$\eta = 4\pi\lambda_p^2/\tau_{es}$$

where τ_{es} is the electron scattering time, then we may write the containment time in the form

$$\tau_c = Rr_t\tau_{ti}/2\lambda_p[1 + R\tau_{ti}/\tau_{es}]^{1/2} \qquad (7)$$

For the special case of electron-ion equilibrium, we obtain for a quiescent hydrogen plasma

$$\tau_c = (RVT)^{1/2}8{\cdot}7 \times 10^{-9}/[1 + 7{\cdot}1 \times 10^2 T^2/RnL]^{1/2} \text{ sec},$$

where $V = \pi r_t^2 L$, is the plasma volume. This has the interesting property that when

$$T^2 = RnL/7{\cdot}1 \times 10^2 \text{ approximately},$$

then τ_c starts to fall off as the temperature is increased. This is because whilst the sound speed in the plasma increases indefinitely, the sheath thickness becomes asymptotic to the plasma wavelength.

In conclusion, we remark that the effect of reversed trapped field is of great importance in freeing the plasma of trapped flux. Experimental observations by KOLB (1959) in particular seem to bear out this argument.

Acknowledgements—The authors would like to thank K. V. ROBERTS of A.E.R.E. Harwell for helpful discussions. One of us (N. J. P) would like to thank Dr. T. S. GREEN of A.W.R.E. Aldermaston for suggestions concerning reversed magnetic fields.

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End loss in the linear θ-pinch

(*Received* 14 *March* 1960)

THE previous letter discussed a θ-pinch in which all the lines of magnetic force that passed through the 'mirrors' at the two ends could be considered as part of an extremely thin current sheath, separating the plasma proper from the main B_z field outside. The plasma trapped within this sheath was assumed either (i) to be essentially field-free, or (ii) to contain only closed magnetic field lines, formed from an initial reversed B_z field in a way which has been explained (WRIGHT and PHILLIPS, previous letter). In either case the main body of the plasma tapered off to a sharp point at the two mirrors, as in the 'cusped configurations' studied by BERKOWITZ *et al.* (1958).

However it seems possible to be faced with an alternative configuration in which some of the magnetic field lines that pass through the mirrors are mixed throughout the plasma. This might happen in two ways:

(a) In some θ-pinch discharges, it is possible to obtain breakdown on the first half-cycle of current, without pre-ionization. During the breakdown phase, (which occupies the first 10 mμ sec or so of the discharge), B_z flux enters the gas, and this is subsequently compressed. Oscillations of such a configuration have been studied by NIBLETT and GREEN (1959).

(b) If breakdown does not take place until the second half-cycle, some 'reversed B_z flux' from the first half-cycle may be trapped within the gas as assumed previously, and then compressed. Mutual cancellation of this reversed field and the main field, at the mirrors, may not be exact so that some field lines from the main body of the plasma still pass through the mirrors.

The loss of plasma from such a configuration will be studied here. For simplicity the current sheath will be treated as infinitely thin. The model adopted is shown in Fig. 1. C is a cylindrical conductor. SS_1 is the coaxial cylindrical current sheath, which encloses a uniform plasma P, containing a uniform axial magnetic field B_i. The ratio of particle pressure p to internal magnetic pressure $B_i^2/8\pi$ is β_0. Outside SS_1 is a vacuum region, containing a uniform axial field B_e.

Beyond the plane Π_1 the flux tube enclosed by the current sheath contracts to a 'throat' or mirror at Π_T, after which it expands again towards Π_2. Since it is only necessary to discuss one of the mirrors the device may be assumed to be of infinite length, i.e. the problem is that of the discharge of a plasma reservoir through an orifice.

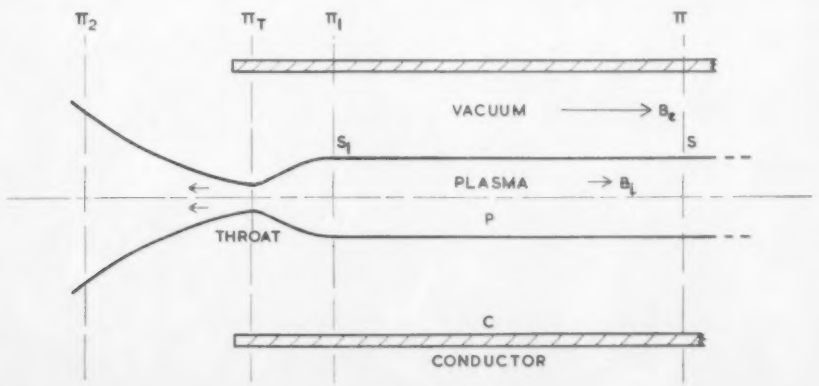

FIG. 1.—Loss of plasma from θ-pinch.

The whole area of the plasma is assumed to be small compared to the area within the conductor C. Therefore by flux conservation

$$B_e(\Pi_T) \cong B_e(\Pi_1) = B_e(\Pi) \tag{1}$$

At the throat Π_T the particle pressure (at any rate in the transverse direction) is taken as zero. Hence from pressure balance at Π_T and Π_1 (or Π), and from (1), it follows that

$$\left(\frac{B_i^2}{8\pi}\right)_{\Pi_T} = \left(p + \frac{B_i^2}{8\pi}\right)_{\Pi_1} \tag{2}$$

i.e. the area of the throat is determined by the fact that the internal magnetic field must be compressed at Π_T until it becomes equal to the external field B_e. Thus

$$\frac{B_i(\Pi_T)}{B_i(\Pi_1)} = \sqrt{1+\beta_0} \tag{3}$$

so that the escape of particles will be slowed down by an internal mirror ratio $R_i = \sqrt{1+\beta_0}$ even if there is no external mirror ratio imposed (as in Scylla (ELMORE, 1958)) by the shape of the conductor.

The θ-pinch is assumed to differ from a classical mirror machine in the following way. An ion will undergo a series of encounters with one or other of the two mirrors. Each time it passes a plane Π_1, its velocity vector will make a certain angle θ with the axis. In a classical mirror machine there is a strong correlation between successive values of θ; they will in fact be equal except for a small rate of diffusion in velocity space. Here it is assumed that *successive values of θ are quite uncorrelated*. This is most likely to be due to collisions at the high particle density used in θ-pinches; it could also result from non-adiabaticity due to a large Larmor radius within the plasma or to a rapid variation of B_i along the axis.

In order to estimate the rate of escape of plasma in such a device, an argument will be used which is half-way between the classical mirror loss theory, and the application of Bernoulli's theorem to magnetohydrodynamics mentioned in the previous letter (equations (1)–(3)). Electrons and longitudinal electric fields will be ignored. Ions are assumed for simplicity all to travel with the same velocity $\bar{v}$; their density in the main body of the plasma is n. They are emitted isotropically at the rate $n\bar{v}\cos\theta/4\pi$ per unit solid angle towards the left from the plane Π_1, afterwards travelling adiabatically and without making any further collisions until they either escape through the throat or return across Π_1.

With this assumption, only ions emitted from Π_1 within the loss-cone in velocity space with semi-angle

$$\theta_c = \sin^{-1}\frac{1}{\sqrt{R_i}} \tag{4}$$

will be lost. The total rate of escape is therefore

$$\frac{n\bar{v}}{4\pi}\int_0^{\theta_c} 2\pi\sin\theta\cos\theta\,d\theta = \frac{n\bar{v}}{4}\frac{1}{R_i}. \tag{5}$$

The rate of escape with no mirror present is $n\bar{v}/4$, so that the effect of the finite plasma pressure β_0 is to reduce it in the mirror ratio $R_i = \sqrt{1+\beta_0}$. (From a hydrodynamic point of view, this is also the ratio of the cross-section of the flux tube at Π_T to that at Π_1). When an external mirror ratio R_e is applied by shaping the conductor C as in Scylla (ELMORE, 1958), the rate of escape is reduced by the total mirror ratio $R = R_i R_e$.

Turning now to a finite cylinder of plasma of length L, (with two ends), we find for the logarithmic rate of loss

$$\frac{1}{n}\frac{dn}{dt} = \frac{n\bar{v}}{4}\cdot\frac{1}{nL}\cdot\frac{1}{R}\cdot 2\ \text{sec}^{-2} = \frac{24{\cdot}7T_p^{\frac{1}{2}}}{RL}\ \mu\ \text{sec}^{-1}$$

for hydrogen, where T_p is the temperature of the proton gas in keV.

This analysis brings out a severe requirement on 'cusped configurations,' if they are to be used for serious plasma containment. *Only a negligible quantity of flux from the main body of the plasma can be allowed to pass through the cusps.* If ions are to be retained while they make N collisions with the cusps, the ratio of this internal field to the external field must not exceed $2/N$.

Acknowledgements—The analysis made here was suggested by results presented by A. C. KOLB (1959) at Uppsala and in a subsequent private discussion. The writer is indebted to the authors of the previous letter and to other members of the A.W.R.E. fusion group for valuable conversations.

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J. Nucl. Energy, Part C: Plasma Physics, 1960, Vol. 1, pp. 245 to 246. Pergamon Press Ltd. Printed in Northern Ireland

BOOK REVIEWS

Handbuch Der Physik. Edited by S. FLUGGE, Vol. XLIV Nuclear Instrumentation—I. Springer–Verlag, Berlin.

THE design and construction of particle accelerators of increasing size and complexity has continued unabated since Lawrence built his first cyclotron in 1930. Yet, until the present volume of the *Handbuch* appeared, there was no comprehensive account of the subject collected together in one volume. Separate chapters are devoted to the cascade generator, Van de Graaff generators, the cyclotron, the betatron, proton synchrotrons and linear accelerators; the final chapter deals with reactor techniques.

Cascade and Van de Graaff generators were amongst the earliest of the accelerating machinery. Today the former conveniently cover the energy range up to one million volts whilst the latter is capable of reaching ten million volts. The chapter on cascade generators (in German) by E. Beldinger is confined to a detailed description of the high voltage set. The chapter on the Van de Graaff generator by R. G. Herb is rather wider in scope. Extensive quotations from personal experience of existing machines is included. Advances in techniques have usually come from direct extrapolation; careful development has slowly overcome causes of voltage breakdown. It is unfortunate that the tandem generator received such brief mention; probably because the articles were completed early in 1958.

Of the four articles on cyclic accelerators, the account of proton synchrotrons by G. K. Green, and E. D. Courant is the most comprehensive. When this chapter was written there were four operating synchrotrons; the Cosmotron at 2,3 Bev; the Birmingham Proton Synchrotron at 1 BeV; the Bevatron at 6,2 BeV and the Moscow Synchrophastron at 10 BeV. After presenting a review of orbital theory the authors give a very excellent description of the four machines drawing attention throughout to points of difference in design, construction and operating characteristics. This chapter also contains a very elegant and condensed treatment of the theory of alternating gradient (AG) synchrotrons. It ends with an account of the Brookhaven AGS and CERN PS. Both machines employ the AG principle and are in the energy range 25 to 30 BeV.

By comparison the companion chapter on electron synchrotrons by R. R. Wilson is sketchy and ill-balanced in parts. Recent work at M.I.T. and Hamburg receives but brief mention.

The chapters on cyclotrons by B. L. Cohen and betatrons by D. W. Kerst deal with machines of longer standing. A brief account of the theory of azimuthally varying field (A.V.F.) accelerators is also given however. Readers should note that there are a few errors in the theory of phase stability in the chapter on cyclotrons.

The article by L. Smith on linear accelerators contains a comprehensive review of both electron and heavy ion machines. The author has conveniently collected together much information which previously was contained in internal reports.

The final chapter on reactor techniques by D. J. Hughes appears to be slightly out of line with the other chapters. It is concerned mainly with the utilization of neutrons either as beams or with internal targets.

In conclusion this volume of the *Handbuch* will be welcomed by accelerator physicists as well as the non-specialist as the most comprehensive account yet given of this field of research.

W. W.

Plasma Physics and Thermonuclear Research: *Progress in Nuclear Energy* Series XI, Vol. I. Pergamon Press 1959, 612 pp., $15 (£5.5.0).

THE Second International Conference on the Peaceful Uses of Atomic Energy, held at Geneva in 1958, forms an important landmark in the development of thermonuclear research. Here was presented, in the short space of four days, the accumulated results of some seven years of research by the major powers, the publication of which had hitherto been prevented by security restrictions. Of the one hundred and eleven papers presented at the conference, and published in full in Volumes 31 and 32 of the Proceedings, nine could be classified as surveys of the progress made and work being carried out in one particular country or institute, four as survey papers on one type of fusion device, some fifty-eight give more detailed accounts of a particular experiment or machine, while forty are theoretical. The book under review contains a selection of thirty-seven of these, or about 40 per cent by volume of the total material presented at the conference. It includes all the survey papers, eleven of the experimental, and thirteen of the theoretical papers.

As is inevitable at a conference of this nature, many of the theoretical papers were highly condensed, making difficult reading for anyone not acquainted with the particular problem considered, while experimental papers, describing in detail the equipment used and results obtained, sometimes overlap with descriptions of similar work performed in another country. The selection contained in this book brings together most of the papers of general interest and those likely to be of more permanent value, while it omits many of the more specialized papers and reduces the duplication which occurred in the full proceedings. The newcomer to the subject will find in the survey papers a clear and comprehensive account of the underlying problems and objectives, while perusal of the more specialized papers will show how these problems are being approached and the progress made in understanding the many complex phenomena involved. The research worker may find missing a few of the papers relevant to his particular subject, but this book should contain most of the material of interest to him. While it is unlikely that many will master the full contents of this volume, it should remain a most valuable reference book for a considerable period, though its use in this respect would have been facilitated by inclusion of the Conference Paper Numbers in the list of contents, the form in which they are usually referred to in scientific journals.

T. E. S.

Project Sherwood—The U.S. Program in Controlled Fusion, AMASA S. BISHOP, Addison-Wesley, 1958. 216 pp., $6.50.

ALTHOUGH Dr. BISHOP's authoritative, historical account of the United States programme in research upon controlled fusion is now well known, being almost two years old (its publication was timed for the 1958 Geneva Conference), it is perhaps still worthy of a short review because its content is of lasting interest. The varied experimental attempts to generate thermonuclear plasmas, and the apparatuses with their curiously uninformative names, are set in relation to one another with a clarity that is perhaps not to be found elsewhere. The growing awareness of the difficulty of the problem is developed, but the book leaves a general impression of open-ended optimism that has perhaps not been fully borne out by subsequent developments. Administrative history is also included, but is segregated so that the reader, like the reviewer, can omit it as desired. In short, the book is a readable, model primer to the subject, and in physical format it is marked by the clarity and beauty of the illustrations. It has reappeared, with a modernizing supplement (to June 1959) as an Anchor Book publication of Doubleday.

A. H. S.

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J. Nucl. Energy, Part C: Plasma Physics, 1960, Vol. 1, pp. 247 to 252. Pergamon Press Ltd. Printed in Northern Ireland

AN EXPERIMENTAL TEST OF THE BEHAVIOUR OF CHARGED PARTICLES IN AN ADIABATIC TRAP*

S. N. RODIONOV

(*Received* 11 *April* 1959)

Abstract—The motion of tritium β-particles in an adiabatic trap is considered. It is shown that under some conditions the β-particles may be reflected from the strong field region (the magnetic mirror) more than 10^7 times. The lifetime of the particle has been studied as a function of the gas pressure and of the configuration and uniformity of the magnetic field.

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1. INTRODUCTION

AN important characteristic of an adiabatic trap is the lifetime of a trapped charged particle. A basic factor determining the lifetime is the Coulomb scattering of the particle. This effect was discussed in detail by BUDKER.[1]† He used an adiabatic approximation, i.e. he assumed that, as a consequence of the conservation of particle magnetic moment μ, every charged particle exists in the trap separately (i.e. without making collisions with other particles) for an arbitrarily long time. In fact since the magnetic moment of a charged particle in a magnetic field is only approximately (adiabatically) invariant it may change slowly. Theoretical estimates of this change[2] only show that for $\rho/R \to 0$ the quantity $\Delta\mu/\mu$ tends to zero faster than any power of ρ/R (ρ is the Larmor radius of the particle, R the radius of curvature of the magnetic line of force). The value of $\Delta\mu/\mu$ at the final value of the ratio ρ/R remains completely indeterminate. At the same time in conditions of practical interest (for instance in devices such as OGRA), the ratio R/ρ is only a few units while the number of reflections from the magnetic mirror must be of the order of 10^5–10^6. For such a large number of reflections the cumulation of even very insignificant changes of μ may lead to the particle escaping from the trap. In particular, azimuthal inhomogeneities of the magnetic field[1] may act in this way. For this reason the experimental study of the behaviour of charged particles in adiabatic traps is absolutely essential.

BISHOP[3] and ANTROPOV, BELYAEV and ROMANOVSKII[4] have attempted to perform such experiments, using external particle injection. However, as a result of the indeterminacies related to the 'detaching' of the particles from the injector, these authors did not succeed in obtaining precise results.

These difficulties were avoided in the present work by making use of the β-particles formed within a tritium-filled trap. The charged particle density did not exceed 10^4 cm^{-3} so that it was possible to neglect collective interactions completely.

2. EXPERIMENTAL ARRANGEMENT

When a β-particle formed by tritium decay moves inside the trap it ionizes the atoms of the gas. The ions and slow electrons so formed can be collected by a weak electric field which does not influence the motion of the β-particles (as is done in an ionization chamber). The magnitude of the so measured ionization current is proportional to the number of β-particles formed in unit time (i.e. to the tritium pressure) and to the number of ion pairs produced by each β-particle. The latter quantity is in turn proportional to the tritium pressure and to the lifetime of the β-particle in the trap.

If the removal of β-particles is determined by their scattering on tritium particles, the lifetime is inversely proportional to the pressure so that the ionization current will be directly proportional to the first power of the tritium pressure. If however there is also some additional mechanism for the removal of β-particles from the trap, then on decreasing the pressure below a certain value the lifetime of the β-particles becomes pressure-independent, so that the ionization current becomes proportional to the square of the pressure. The experiment therefore consisted in measuring the dependence of ionization current on tritium pressure. From the magnitude of the pressure at which the transition from quadratic dependence to linear dependence takes place, one can determine a lifetime for the β-particle in the adiabatic trap that is not determined by scattering on atomic tritium.

* Translated by N. KEMMER from *Atomnaya Energiya* **6**, 623 (1959).
† In this work only binary collisions were considered. The question of the influence of collective interactions on the loss of particles from a trap has so far not been resolved.

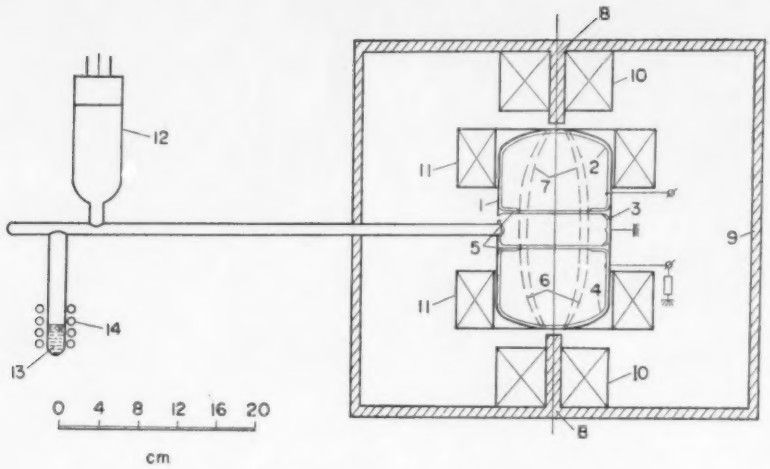

FIG. 1.—Schematic section of the apparatus.
(1) Glass chamber; (2) High voltage electrode; (3) Guard ring; (4) Collector; (5) Annular glass projections; (6) Limit of the working volume for slow ions and electrons; (7) Limit of the working volume for fast β-particles; (8) Electromagnet poles; (9) Yoke; (10 and 11) Coils; (12) Ionization gauge; (13) UT_3 heater.

3. DESCRIPTION OF THE APPARATUS

The magnetic field of the trap was produced by means of a yoke type electromagnet (19) (Fig. 1) with additional windings (11). The maximum field near the poles H_m was equal to 5000 oersted while the field at the centre of the system was $H_0 = 600$ oersted. The mirror ratio $H_m/H_0 = \gamma$ could vary within the limits 1–40.

A glass chamber (1) was placed between the poles of the magnet with aluminium electrodes (2)–(4) deposited on its interior surface. The chamber was subjected to preliminary baking while being evacuated at a temperature of 400–450°C lasting several days, after which it was sealed off from the vacuum system. Then the chamber was joined to the tube (13), which contained a sample of freshly prepared uranium tritide UT_3 (with a small admixture of ^{3}He). The tritium activity of the sample was about 1 curie. The tube (13) also contained some uranium powder which served as a getter for air, the residual pressure of which was of the order of 10^{-7} mm Hg.* As the sample of UT_3 warmed up, the tritium pressure in the chamber rose (up to $\sim 10^{-3}$ mm Hg at $\sim$100°C) and as the sample was cooled it fell (down to $\sim 10^{-7}$ mm Hg at the temperature of liquid nitrogen).

The gas pressure in the chamber was measured with standard ionization gauges graduated with tritium. The calibration was performed by measuring the current $^3He^+$ ions formed as a result of the β-decay of tritium in the absence of the trap field. (When there are no trapped particles the ion current due to ionization of the gas is unmeasurably small at these pressures.) The ionization gauge served at the same time as a pump for removing the ^{3}He.

The measurement of ionization current was performed with a standard electrometer amplifier with a sensitivity threshold of 5×10^{-14} A. Currents from 5×10^{-13} up to 5×10^{-9} A were measured. The source of voltage was a battery placed inside a screened box. All connexions were by means of screened leads. The resistance across the glass between the collector and the guard ring was $\sim 10^{11}\ \Omega$, and did not shunt the input resistance of the amplifier (10^8–$10^{10}\ \Omega$).

4. DETERMINATION OF BACKGROUND CURRENTS

There are three main sources of background currents: the current of $^3He^+$ ions; secondary electron emission from the cathode under the action of the β-particles leaving the chamber walls; and secondary emission under the action of β-particles impinging on the cathode from the chamber volume.

In the presence of a magnetic field the current of $^3He^+$ ions is produced only by those ions which are formed within the volume (6) (see Fig. 1) that is bounded by the lines of force which touch the internal annular baffles (5) in the chamber. This volume can be calculated from the known magnetic field configuration.

In the course of the experiments it was noticed that a considerable quantity of tritium ($\sim$10 mc) was deposited on the chamber wall. The time of deposition was several days, so that in the course of a separate experiment lasting several hours, the tritium concentration on the walls could be assumed constant. The β-particles produced by the decay of this tritium

* The author wishes to express his sincere thanks to M. D. SENIN and YU. V. GAGARINSKII for preparing the sample of UT_3 and adapting the chamber for the experiment.

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were not captured in the trap, but collided with the walls after several revolutions, so producing secondary electron emission. Because of the form chosen for the electrodes (2) and (4) (see Fig. 1) the background current produced only those secondary electrons which appear on the region of the cathode adjacent to the volume (6). This current was measured by freezing out the tritium ($p_T < 10^{-7}$ mm Hg).

The β-particles formed in the volume (6) and incident on the cathode also caused secondary emission and a background current which was proportional to the tritium pressure. To evaluate this component of background current the coefficient of secondary emission was required. It was determined by special measurements and proved to be equal to 1.

In all subsequent measurements the background currents were subtracted.

5. CONTROL MEASUREMENTS

The dependence of the ionization current on the collecting voltage was measured (Fig. 2). A characteristic diminution of current at high voltages (>750 V) was found, at which the electric field begins to influence the motion of the β-particles. The main measurements were performed at a collecting voltage of ~200 V. The background currents remained constant at high collecting voltages, confirming the absence of discharges in the system.

In order to verify whether the results are influenced by surface charges on the chamber glass the sign of the voltage on the h.t. electrode was regularly changed. In all cases the current changed its sign but remained unchanged in absolute magnitude.

The ionization current in a uniform magnetic field ($\gamma = 1$) was also measured. Within experimental error it proved to be zero, showing that background currents were being taken into account correctly.

6. BASIC RESULTS

If there is another gas in addition to tritium in the trap the ionization current on the collector (4), (Fig. 1) depends on the partial pressures in the following manner

$$I \sim \frac{p_T(p_T + \alpha p_f)}{p_T + p_0}.$$

Here p_T is the tritium pressure, p_f is the pressure of foreign gas; p_0 is the tritium pressure at which the removal of β-particles from the trap due to scattering on tritium atoms is equal to the removal for other reasons (including removal by scattering on the foreign gas). $\alpha = \sigma_{if}/\sigma_{iT}$ (σ_i being the ionization cross-section).

The current in the ionization gauge is $I_m \sim (p_T + \alpha p_f)$ with the same value of α as in the expression for I, because at an electron energy greater than 200 V (the anode voltage of the ionization gauge) α is practically independent of electron energy.

Thus the ratio of the ionization currents in the trap and in the gauge is proportional to $p_T/(p_T + p_0)$. At $p_T = p_0$, this ratio is halved compared to its value for large p_T. An example of how p_0 is determined by the method described is given in Fig. 3.

The important foreign gas in the present series of experiments was ^{3}He. Almost the whole tritium activity was contained in the UT_3 (not more than 10 per cent of the tritium passed into the chamber). Therefore the ^{3}He formed was in the main retained in the UT_3 crystal lattice and when the sample was heated the helium was liberated from the sample together with the tritium in a quantity proportional to the latter. The coefficient of proportionality was determined by special experiments involving the freezing out of tritium. Knowing the vapour pressure of ^{3}He one can find the tritium pressure from the readings of the ionization gauge.

We consider first the measurement of the β-particle lifetime in the trap due to their scattering by tritium atoms ($p_T \gg p_0$). It is sufficient for this purpose to know the energy loss of each β-particle in the trap or the number of ion pairs produced by the β-particle.* This number is equal to the ratio of the ionization current to twice the current of ^{3}He ions† from the volume bounded by the magnetic lines of force which are at a distance $\bar{\rho}$‡ inside the internal annular baffles in the chamber (see (7) on Fig. 1). ($\bar{\rho}$ is the Larmor radius of the β-particle in the magnetic field H_0, averaged over the β-spectrum.) The working volume must be

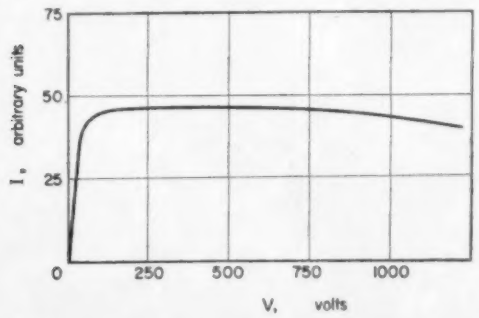

FIG. 2.—Dependence of the ionization current I on the collecting voltage V.

* It was assumed that the production of 1 ion pair required on the average 60 eV.[5]

† The $^3He^+$ ion current had to be doubled owing to the fact that the β-particles impinged on the anode and the cathode with equal probability and did not produce a current in the measuring circuit.

‡ The β-particles formed outside this volume perform several revolutions and then impinge on the annular baffles of the chamber.

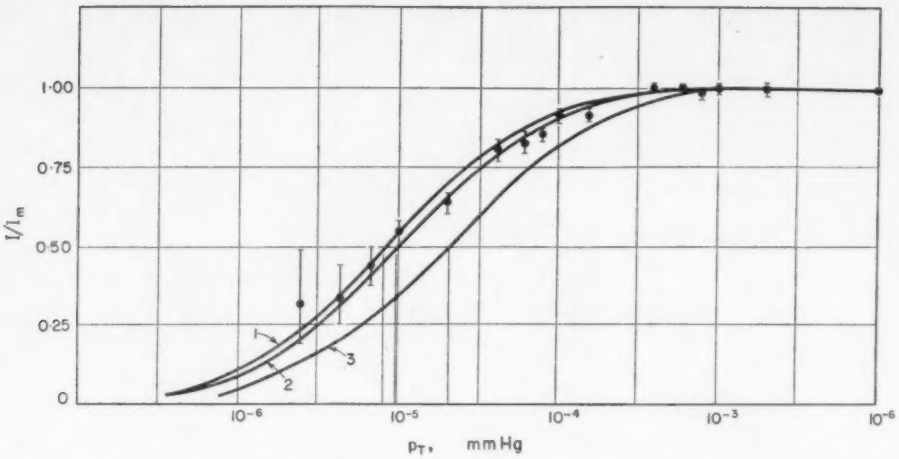

FIG. 3.—Determination of p_0.
The curves represent the function $p_T/(p_T + p_0)$. (1) $p_0 = 8 \times 10^{-6}$ mm Hg; (2) $p_0 = 10^{-5}$ mm Hg; (3) $p_0 = 2 \times 10^{-5}$ mm Hg.

determined separately for each configuration of the magnetic field.

Knowing the energy lost by a β-particle using the usual formula for ionization losses, one can calculate the lifetime of the β-particle in the trap and also the path length it travels and the number of reflections from the magnetic mirrors. To calculate the number of reflections one calculates the mean path lengths of the β-particle between reflections. According to the configuration of the magnetic field this quantity varies between 15 and 20 cm. The relation between ΔE, the energy loss of the β-particle in the trap, the number of reflections N and the lifetime τ is shown in Table 1. (It was assumed that the mean initial energy of the tritium β-particles was 6 keV.)

TABLE 1.—ENERGY LOSS, NO. OF REFLECTIONS AND LIFETIMES OF β-PARTICLES IN THE TRAP

ΔE (keV)	$N \cdot p_T$ (mm Hg)	$\tau p_T \times 10^8$ (sec mm Hg)
1	5·7	4
2	11·1	8·3
3	15·5	12
4	18·3	15
5	20·6	18·3
6	21·4	20·2

The dependence of τ, the β-particle lifetime in the trap (in units of the time of stopping by ionization, which is $2 \times 10^{-7} \times p_T^{-1}$ sec) on the different parameters of the trap for $p_T \gg p_0$, is given in Figs. 4–6.

Fig. 4 gives the dependence of τ on the magnetic field H_0 for $\gamma = 5$. Each point represents the mean of six measurements performed at $p_T \sim 10^{-3}$ mm Hg. The errors indicated (as also in the other figures) are statistical errors.

Fig. 5 gives the dependence of τ on H_0 for given H_m. A pronounced characteristic of this dependence is the presence of the maximum of τ predicted by BUDKER.[1] This maximum is connected with the fact that at small H_0 the removal of β-particles across the magnetic field becomes greater, while at large H_0 the removal through the mirrors (owing to the diminution of λ) increases. The position of the maximum agrees with theoretical estimates.

Fig. 6 gives the dependence of τ on γ for $H_0 = 300$ oersted. According to BUDKER this dependence is proportional to the function

$$\psi(\gamma) = -\ln \tan \varphi_0/2 - \cos \varphi_0; (\sin^2 \varphi_0 = 1/\gamma).$$

For β-particles formed in different regions of the chamber $\gamma = H_m/H$ has different values and is not constant as was assumed for simplicity by BUDKER. Therefore an average of $\psi(\gamma)$ was taken over all β-particles of different γ. The calculated curve so obtained was matched to the experimental points at large γ. If one formally uses BUDKER's results one obtains a somewhat low value for the mean β-particle energy of the matched calculated curve (~1·5 keV).

In order to find the N and τ that correspond to the external mechanism of β-particle removal from the trap (other than scattering on tritium atoms) one can use the data of Table 1 with p_T replaced by p_0. As was already remarked the main such mechanism is the scattering of β-particles on ^{3}He atoms. It was found in the experiments that the dependence of p_0 on p_{He} can be expressed as follows:

$$p_0 = a p_{He} + b.$$

The residual value of p_0, namely b, may be attributed to the presence of traces of air in the trap and

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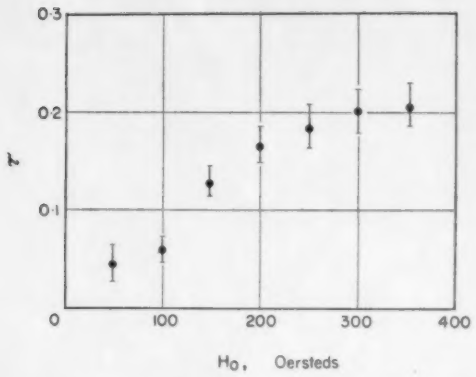

FIG. 4.—Dependence of the lifetime of the β-particle in the trap on the magnitude of the magnetic field at the centre H_0 for $\gamma = 5$.

to the possible change of magnetic moment of an individual non-colliding, β-particle (without collisions).

For certain configurations of magnetic field ($H_0 >$ 200 oersted, $\gamma < 15$) the value of b remained below 2×10^{-6} mm Hg. Assuming that there is no air in the trap one can get a lower limit for the number of reflections from the magnetic mirror, related to the change of μ for an individual β-particle. In particular for $H_0 = 400$ oersted, $H_m = 4500$ oersted, we get $N > 10^7$ and $\tau > 0{\cdot}07$ sec. (In these conditions the β-particle forms 65 ion pairs and loses about 4 keV of energy before escaping from the trap.)*

For some configurations of magnetic field the value of b found was considerably greater than 2×10^{-6} mm Hg, which can only be explained by a change of the μ of an individual β-particle. Thus, for instance, if the magnetic field H_0 was reduced to 90 oersted and γ increased to 40, the removal of β-particle from the trap was greatly increased ($N = 2 \times 10^5$).

The removal of β-particles from the trap was also increased if an azimuthal asymmetry was introduced. In normal conditions the azimuthal asymmetry did not exceed 0·5 per cent at the centre of the trap, and 4 per cent near the poles. An artificial inhomogeneity was produced by means of a coil placed outside the chamber in the median plane of the magnetic field. In all cases the introduction of azimuthal asymmetry produced a decrease of the ionization current which was apparently connected with the change of the working volume in the trap. The results of measurements in the presence of azimuthal asymmetry are given below (Table 2). (The magnitude of the additional azimuthal asymmetry $\Delta H/H$ is indicated.)

TABLE 2

H_0	H_m	γ	$\Delta H/H$ (%)	N
300	4000	13	0	$>6 \times 10^6$
			3	$>6 \times 10^6$
90	3600	40	0	2×10^5
			9	2×10^4
			15	2×10^4
200	960	5	0	$>3 \times 10^6$
			4	$>3 \times 10^6$

7. CONCLUSIONS

The method used has enabled us to obtain quantitative data on the behaviour of charged particles in an adiabatic trap. This is of practical interest in the discussion of all traps with magnetic mirrors currently being considered.

The β-particle lifetime in the trap related to Coulomb scattering on the gas atoms is in satisfactory agreement with the theory of BUDKER,[1] being valid for up to 10^7 reflections from the magnetic mirrors. The influence of

* As the scattering cross-section for β-particles by air atoms is $Z(Z+1)/2$ times greater than by tritium atoms the residual air pressure necessary to explain the observed value of b is only $\sim 7 \times 10^{-8}$ mm Hg. In our experiments the partial air pressure was not directly measured.

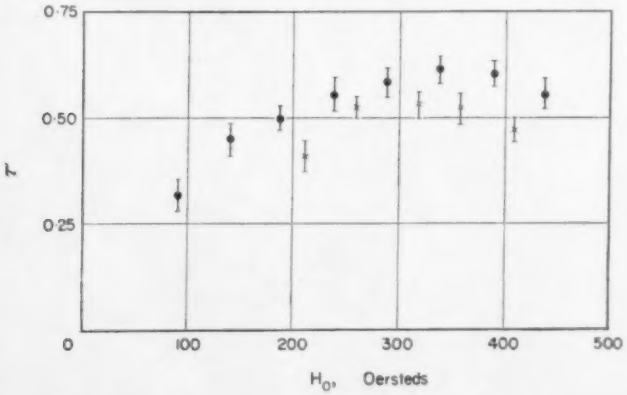

FIG. 5.—Dependence of the lifetime of the β-particles in the trap on the magnitude of the magnetic field in the centre H_0 for $H_m = 4000$ oersted (●), $H_m = 3000$ oersted ×.

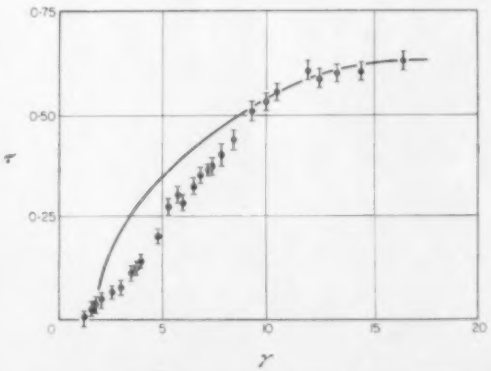

FIG. 6.—Dependence of the β-particle lifetime τ on γ. (The solid curve is theoretical.)

azimuthal inhomogeneities which we have demonstrated apparently does not contradict the results of calculations performed very recently by CHIRIKOV.[6]

Acknowledgements—The author would like to take this opportunity to express his deep gratitude to G. I. BUDKER for his constant attention and advice and for initially suggesting this work and also to P. E. SPIVAK who gave much help in the initial stages of the experiments.

Note added in proof: It has become known to us that similar experiments were performed by G. GIBSON and E. LAUER, *Bull. Amer. Phys. Soc.* **3,** 412 (1958). Details of this work have not yet been published.

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RESONANCE PROCESSES IN MAGNETIC TRAPS*

B. V. CHIRIKOV

Abstract—Consideration is given to resonances between the Larmor rotation of charged particles and their slow oscillations along the lines of force. Under certain conditions these resonances can result in a complete exchange of energy among the degrees of freedom of the particle, so that the particle escapes from the trap. The influence of resonances on adiabatic processes associated with a time variation of the magnetic field is also examined.

1. INTRODUCTION

ONE of the methods for thermally insulating a plasma in order to realize a controlled thermonuclear reaction is the use of so-called adiabatic traps, or traps with magnetic mirrors, proposed and calculated by BUDKER.[1] Similar systems have been proposed by YORK[2] and calculated by JUDD, MCDONALD and ROSENBLUTH.[3] Recently, considerable developments in this direction have occurred and therefore it is of interest to study further similar systems.

The action of an adiabatic trap is based[1] on the conservation of orbital magnetic moment of a charged particle in a magnetic field ($\mu = Mv_\perp^2/(2H)$ where $v_\perp$ is the component of the particle velocity in a direction perpendicular to the magnetic field H). It is a necessary, but of course not a sufficient condition for the usefulness of a trap that it can entrap a single charged particle. Generally speaking the lifetime of such a particle in the trap is not infinite because the magnetic moment is only an adiabatic invariant, i.e. it can change slowly and so allow a redistribution of energy among the longitudinal and transverse degrees of freedom of the particle and consequent escape from the trap.

The question of the time variation of an adiabatic invariant has been considered in a number of papers.[6–8] However, only KULSRUD[7] takes his calculations as far as concrete results for a harmonic oscillator, obtaining

$$\frac{\Delta I}{I} = \frac{2\,\Delta^{(q)}}{(2\omega_0)^{q+1}} \cdot \cos\left(2\theta_0 - \frac{\pi q}{2}\right) \qquad (1.1)$$

Here I is the adiabatic invariant, $\Delta^{(q)}$ is the discontinuity in the qth derivative of $\omega(t)$, θ_0 and ω_0 are the phase and the frequency of the oscillator at the time of the discontinuity in the derivative. The basically unsatisfactory feature of the above expression is its asymptotic nature. This means that it is correct only if $1/(\omega T) \to 0$ (T being the characteristic time for the variation $\omega(t)$. For finite values of the adiabaticity parameter $1/(\omega T)$ equation (1.1) is not always correct. (The conditions for its applicability are given in the Appendix.) In the particular case where $\omega(t)$ is an analytic function, equation (1.1) gives $\Delta I/I = 0$. This means that when $1/(\omega T) \to 0$ the quantity $\Delta I/I$ tends to zero faster than any power of the parameter $1/(\omega T)$ (for instance as $\exp(-\omega T)$), but it remains unknown how exactly it behaves. For this reason the normally used methods of asymptotic expansion in powers of a small parameter such as $(1/\omega T)$ are not applicable in this case.

In the present paper we consider a different approach to this problem. It is based on the simple physical model of resonances between the Larmor rotation of the charged particle and slow oscillations of the particle along the magnetic lines of force.† Such resonances are possible in spite of the differences in frequency if the slow oscillations of the particle are anharmonic and contain high harmonics of their basic frequency. The action of the resonances leads in particular to a change in the magnetic moment of an individual particle (ignoring collisions).

2. BASIC EQUATIONS

The present paper does not aim to produce formulae for computation. The main attention is directed to the physical processes taking place when a charged particle moves in a magnetic trap. We therefore confine ourselves to the study of the simple Hamiltonian used by FIRSOV[6] ($M = 1$)

$$\mathscr{H} = \frac{p_x^2 + p_y^2 + \omega^2(x)y^2}{2}\,;\; p_x = \dot{x};\; p_y = \dot{y}. \qquad (2.1)$$

Here x and y are the co-ordinates along and across the magnetic line of force respectively and ω is the Larmor frequency. The equations of motion have the form

$$\ddot{y} = -\omega^2 y;\quad \ddot{x} = -\omega\frac{d\omega}{dx}\,y^2. \qquad (2.2)$$

* Translated by N. KEMMER from *Atomnaya Energiya* **6**, 630 (1959).

† The importance of resonances for the change of adiabatic invariants has been pointed out by ANDRONOV, LEONTOVICH and MANDEL'SHTAM.[9]

Since the oscillations along the x-axis are slow ($\Omega \ll \omega$) the solution for y can be stated in the form

$$y = \rho \cos\theta; \quad \theta = \int \omega\, dt + \varphi. \tag{2.3}$$

Here ρ is the Larmor radius of the particle and

$$\ddot{x} = -\omega \frac{d\omega}{dx} \frac{\rho^2}{2} (1 + \cos 2\theta).$$

Since $\omega\rho^2/2 = I$, where I is the adiabatic invariant in which we are interested and which is connected to the magnetic moment by the relation $I = c\mu/e$, we obtain

$$\ddot{x} + I \frac{d\omega}{dx} = -I \frac{d\omega}{dx} \cos 2\theta. \tag{2.4}$$

Thus the motion along the x-axis is oscillatory with a potential energy $I\omega(x) = \mu H(x)$ and superimposed on it a fast periodic perturbation of frequency 2ω. This perturbation is usually neglected because $\omega \gg \Omega$. However, if the oscillations along the x-axis contain high harmonics of the basic frequency, a resonance is possible between the high frequency perturbation $I(d\omega/dx)\cos 2\theta$ and one of these harmonics.

The effect described can also be approached from a different direction. Let us consider the equation $\ddot{y} + \omega^2(t)y = 0$ in which the dependence of ω on time is related to the oscillations along the x-axis. The period of the function $\omega(t)$ is much greater than $1/\omega$, but if $\omega(t)$ contains high frequencies right up to ω, one of these may produce a parametric resonance.

Since $\mathscr{H}$, the total energy of the particle, is conserved, both the resonances mentioned lead to a redistribution of the particle energy among its degrees of freedom. To investigate these resonances, we shall use a method described elsewhere by the author.[10] First we introduce the Hamiltonian $\mathscr{H}_y$ which describes the motion along the y-axis

$$\mathscr{H}_y = \frac{p_y^2 + \omega^2(t)y^2}{2}. \tag{2.5}$$

Then $d\mathscr{H}_y/dt = \partial\mathscr{H}_y/\partial t = \omega\dot{\omega}y^2$. For the variation of the adiabatic invariant $I = \mathscr{H}_y/\omega$ we find

$$\frac{dI}{dt} = \dot{\mathscr{H}}_y/\omega - \mathscr{H}_y\dot{\omega}/\omega^2 = \dot{\omega}(y^2 - \overline{y^2}), \tag{2.6}$$

where the bar denotes an average over the phase which changes with the frequency ω. Considering ω as a parameter, we obtain a correction to the frequency as $\dot{\varphi}$ where φ is given by (2.3)[10]

$$\dot{\varphi} = -\frac{\omega\dot{\omega}}{\dot{y}} \left[\omega y^2 \left(\frac{\partial y}{\partial I}\right)_{\omega,\theta} + \left(\frac{\partial y}{\partial \omega}\right)_{I,\theta} \right] = -\frac{\dot{\omega}}{2\omega} \sin 2\theta. \tag{2.7}$$

We also introduce the Hamiltonian $\mathscr{H}_x$ which describes the motion along the x-axis as given by (2.4)

$$\mathscr{H}_x = \frac{p_x^2}{2} + I\omega(1 + \cos 2\theta). \tag{2.8}$$

Putting

$$x = x(I,\vartheta); \quad \vartheta = \int \Omega(I)\, dt + \psi \tag{2.9}$$

and taking into account that the quantity $W_x = p_x^2/2 + I\omega$, is equal to the total Hamiltonian $\mathscr{H}$ and is conserved, we get

$$\dot{W}_x = \frac{\partial W_x}{\partial t} + [W_x, \mathscr{H}_x] = \dot{I}\omega - \dot{x}\frac{d\omega}{dx} I \cos 2\theta = 0,$$

Here [,] denotes a Poisson bracket. Hence

$$\dot{I} = I \frac{\dot{\omega}}{\omega} \cos 2\theta, \tag{2.10}$$

in agreement with (2.6). In analogy with (2.7) we obtain for the $\dot{\psi}$ of (2.9)[10]

$$\dot{\psi} = \frac{\Omega}{\dot{x}} \left(\frac{\partial x}{\partial I}\right)_\vartheta \dot{I} = I \left(\frac{\partial x}{\partial I}\right)_\vartheta \frac{\Omega}{\omega} \frac{d\omega}{dx} \cos 2\theta. \tag{2.11}$$

We note that the equations (2.7), (2.10) and (2.11) are exact for the particular Hamiltonian (2.1) we have chosen.

3. FIRST ORDER RESONANCES

We integrate (2.10) by expanding the right-hand side into a Fourier series. The function $\cos 2\theta$ expresses the frequency modulated oscillation

$$\left.\begin{aligned} \dot{\theta} &= \omega + \dot{\varphi} = \bar{\omega} + \dot{\varphi} + \sum_n \omega_n \cos 2n\vartheta, \\ \theta &= \varphi + \bar{\omega}t + \sum_n \frac{\omega_n}{2n\Omega} \sin 2n\vartheta. \end{aligned}\right\} \tag{3.1}$$

Here, and in the following, the bar denotes an average over phases which change with the frequency Ω. Performing some simple transformations we find

$$\cos 2\theta = \tfrac{1}{2} \sum_n (F_{1n} \pm F_{2n}) \cos 2(\bar{\omega}t + \varphi \pm n\vartheta). \tag{3.2}$$

Here we sum over the two sign combinations, with the upper and the lower signs taken together. The fact that we expand $\omega(t)$ only in cosines is related to the symmetry of the process with respect to the two points of reversal ($\dot{x} = 0$). The factor 2 in (3.1) characterizes the symmetry of $\omega(x)$ relative to the median plane of the magnetic field. The Fourier coefficients F_{1n} and F_{2n} are determined by the equations

$$\left.\begin{aligned} \cos\left(\sum_n \frac{\omega_n}{n\Omega} \sin 2n\vartheta\right) &= \sum_n F_{1n} \cos 2n\vartheta, \\ \sin\left(\sum_n \frac{\omega_n}{n\Omega} \sin 2n\vartheta\right) &= \sum_n F_{2n} \sin 2n\vartheta. \end{aligned}\right\} \tag{3.3}$$

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Now let

$$\frac{\dot{\omega}}{\omega} = \sum_m \Omega_{1m} \sin 2m\vartheta. \tag{3.4}$$

Multiplying (3.2) and (3.4) we obtain the equation

$$\frac{dI}{dt} = I \sum_{\bar{\omega} \approx l\Omega} P_l \cos \psi_l. \tag{3.5}$$

The condition $\bar{\omega} \approx l\Omega$ shows that of the whole sum one should keep only the one term whose frequency is close to zero. This is just the resonance term which gives the greatest contribution to the variation of I.

$$\psi_l = 2(\bar{\omega}t + \varphi_1 - l\vartheta);\ \varphi_1 = \varphi - \pi/4, \tag{3.6}$$

$$\begin{aligned} 4P_l = &- \sum_{m-n=l} \Omega_{1m}(F_{1n} + F_{2n}) \\ &+ \sum_{n-m=l} \Omega_{1m}(F_{1n} + F_{2n}) \\ &- \sum_{m+n=l} \Omega_{1m}(F_{1n} - F_{2n}). \end{aligned} \tag{3.7}$$

Equation (3.5) must be supplemented by an equation for the $\dot{\psi}_l$ of (2.7) and (2.11)

$$\dot{\psi}_l = 2\left(\bar{\omega} - l\Omega - \frac{\dot{\omega}}{2\omega} \sin 2\theta - I \frac{\partial x}{\partial I} \frac{d\omega}{dx} \frac{\Omega}{\omega} \cos 2\theta\right).$$

Taking

$$2I \frac{\partial x}{\partial I} \frac{d\omega}{dx} \frac{\Omega}{\omega} = \sum_m \Omega_{2m} \cos 2m\vartheta, \tag{3.8}$$

we obtain

$$\frac{d\psi_l}{dt} = 2(\bar{\omega} - l\Omega) - \sum_{\bar{\omega} \approx p\Omega} Q_p \sin \psi_p, \tag{3.9}$$

where Q_p is determined by the expression

$$\begin{aligned} -4Q_p = &\sum_{m-n=p} (F_{1n} + F_{2n})(\Omega_{2m} + \Omega_{1m}) \\ &+ \sum_{n-m=p} (F_{1n} - F_{2n})(\Omega_{2m} - \Omega_{1m}) \\ &+ \sum_{m+n=p} (F_{1n} - F_{2n})(\Omega_{2m} + \Omega_{1m}). \end{aligned} \tag{3.10}$$

4. HIGHER-ORDER RESONANCES

The equations (3.5) and (3.9), which we will discuss further below, are approximate not only because we have discarded non-resonance terms in them (these only lead to small oscillations of I) but mainly because not all resonances have been taken into account. In deriving (3.5) and (3.9) we started from (3.1) and (3.2) assuming φ and ψ to be constant, while in fact these quantities contain small periodic components as shown by (2.7) and (2.11). It can be shown that this leads to additional resonances determined by the condition $k\bar{\omega} = l\Omega$ ($k \geqslant 2$). (We shall call k the order of the resonance.) There are also other effects which lead to resonances of higher orders, which we shall not take into account exactly. However, an examination of particular cases has shown that higher order resonances are described by equations similar to (3.5) and (3.9) with the replacements $\bar{\omega} \to k\bar{\omega}$; $P_l \to P_l^{(k)} \simeq P_l \alpha_k$ and $Q_l \to Q_l^{(k)} \simeq Q_l \alpha_k$ where α_k is taken to be equal to the greater of the quantities $(\rho/R)^{2(k-1)}$ and $(Q_l/\Omega)^{k-1}$ (R is the radius of curvature of the magnetic line of force). In the first case, in which ρ/R is involved, both integral and half-integral values of k are possible, while in the second case, involving Q_l/ω, only integral values are allowed. Of course, the rule just formulated is only valid for rough estimates and the whole question requires further study.

5. THE STATIONARY CASE

We shall call conditions stationary if in (2.1) ω is not explicitly dependent on time ($\partial\omega/\partial t = 0$), while in the non-stationary case we have $\omega = \omega(x,t)$.* The stationary case corresponds to a magnetic field which is constant in time and axially symmetric. Non-stationary conditions arise both as a result of azimuthal inhomogeneities† (owing to particle drift) and also when the magnetic field depends explicitly on time.

It is well known (see for instance BOGOLYUBOV and MITROPOL'SKII[11]) that in the present case equations (3.5) and (3.9) determine regions of instability at $\bar{\omega} = l\Omega$ whose widths are $\Delta_r(\bar{\omega} - l\Omega) \sim Q_l$. As will be shown below there is no need to investigate these regions in detail. It is important merely to note that they do not overlap, i.e. that their width Q_l is less than the distance 2Ω between them. This follows directly from the estimate (1.2)‡ for Q_l which gives $Q/\Omega \leqslant \sqrt{(\Omega/\bar{\omega})} \ll 1$.

Higher-order resonances do not change this last inequality, since the total resonance width is

$$\sim \sum_k k Q_l^{(k)} \sim Q_l \sum_k k\alpha_k \sim Q_l;\ (\alpha_k \ll 1).$$

(See Section 4).

However in that case the regions of instability play no part at all because of the non-linearity of the oscillations, or more precisely because of the dependence of the frequencies $\bar{\omega}$ and Ω on I. Even for particles which fall into the unstable regions, I will not change monotonically, but will perform oscillations round its resonance value.[10] In the following

* The time dependence $\omega(t)$ which occurred above was not explicit in x.

† Strictly speaking the Hamiltonian (2.1) is only correct in the axially-symmetric field. However, if change in the magnetic field due to particle drift is small in a time $1/\Omega$ one can retain the form (2.1) with an explicitly time-dependent ω.

‡ (1.1)–(1.3) are the formulae of the Appendix.

we shall call such oscillations phase oscillations in analogy with the phase oscillations of charged particles in accelerators. They can be investigated with the aid of the equation which is obtained from (3.5) and (3.9) by elimination of I:

$$\frac{d^2\psi_l}{dt^2} = 2\frac{\partial}{\partial t}(\bar{\omega} - l\Omega) + 2IP_l\frac{\partial}{\partial I}(\bar{\omega} - l\Omega)\cos\psi_l. \quad (5.1)$$

Here and in the following, we neglect terms in Q_l which is permissible under the condition $Q_l^2 \ll |IP_l\,\partial(\bar{\omega} - l\Omega)/\partial I|$ which is always satisfied.[10] The amplitude of the frequency change in phase oscillations is

$$\Delta_{ph}(\bar{\omega} - l\Omega) \simeq 2\sqrt{\left|IP_l\frac{\partial}{\partial I}(\bar{\omega} - l\Omega)\right|}, \quad (5.2)$$

while the frequency of the phase oscillations is given by

$$\Omega_{ph}^2 \simeq \left|2IP_l\frac{\partial}{\partial I}(\bar{\omega} - l\Omega)\right|. \quad (5.3)$$

The case requiring further special discussion is when the amplitude (5.2) becomes comparable or exceeds the distance between the resonances 2Ω (see Section 7).

6. THE NON-STATIONARY CASE (FAST PASSAGE THROUGH RESONANCES)

If the frequencies $\bar{\omega}$ and Ω are explicitly time-dependent and if the difference $\bar{\omega} - l\Omega$ varies, a resonance is crossed and thus a change of I occurs. We first consider a fast passage through resonance, in which the rate of change of $\bar{\omega}$ and Ω due to the change in I can be neglected in comparison with the rate of change due to the explicit dependence on time.

$$A = \left|\frac{IP_l\dfrac{\partial}{\partial I}(\bar{\omega} - l\Omega)}{\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)}\right| \ll 1. \quad (6.1)$$

In this case the equations (3.5) and (3.9) can be integrated immediately and in first approximation (in $\Delta I/I$) we have

$$\Delta I = \sqrt{\pi}I\sum_{\omega \approx l\Omega}\frac{P_l\cos(\psi_{0l} + \pi/4)}{\sqrt{\left|\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right|}}. \quad (6.2)$$

Here ψ_{0l} is the value of the phase ψ_l at the instants of resonance, ($\dot{\psi}_s = 0$). If the distribution of the phases ψ_{0l} in the different passages through resonance is random we have $\overline{\Delta I} = 0$, and

$$\overline{(\Delta I)^2} = \frac{\pi}{2}I^2\sum_l\frac{P_l^2}{\left|\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right|}. \quad (6.3)$$

Here the summation extends over all resonances crossed and the bar denotes an average over the phases ψ_{0l}. In the next approximation we have

$$\overline{\Delta I} = \frac{1}{2}\frac{\partial}{\partial I}\overline{(\Delta I)^2} + \\ + \frac{\pi}{4}\sum_l\frac{IP_l}{\left|\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right|}\left[Q_l - \frac{\partial}{\partial I}(P_lI)\right]. \quad (6.4)$$

The first moment (6.4) characterizes the systematic change of I while the second moment (6.3) characterizes its spread. Knowledge of the two moments is sufficient to establish Fokker-Planck type equations which can be solved to find the flux of particles into the forbidden cone.[1] However, as is shown by the expressions for the moments, this equation proves very complicated. We therefore consider a simpler method for estimating the change in I.

It follows from (6.4) that $\overline{\Delta I}/I \sim \overline{(\Delta I)^2}/I^2$. If $\Delta I/I \ll 1$ the influence of the first moment may be completely neglected ($[\overline{(\Delta I)^2}/I^2]^{1/2} \gg \overline{\Delta I}/I$). If $\Delta I \gtrsim I$ the influence of the two moments is of equal order of magnitude. For estimates it is therefore sufficient to investigate only the change of $\overline{(\Delta I)^2}$. The condition for escape from the trap then has the form $\overline{(\Delta I)^2} \simeq (I_0 - I_k)^2$, where I_0 is the initial value of I and I_k its value on the surface of the forbidden cone.

Since the number of resonances crossed in unit time is $(k/\Omega)\left|\partial(\bar{\omega} - l\Omega)/\partial t\right|$ we get, in view of (6.3)

$$\frac{d}{dt}\overline{(\Delta I)^2} = \frac{\pi}{2}\frac{kI^2(P^{(k)})^2}{\Omega}. \quad (6.5)$$

This ordinary differential equation is easy to solve either numerically or by approximate methods. Of course, for (6.5) to be valid it is necessary that the change of I in crossing a single resonance should be small compared to I itself.

The coefficient k in (6.5) denotes the order of the resonance crossed and is determined by the range of the variation of $\bar{\omega} - l\Omega$ due to the explicit time dependence.

$$k = \Omega/\Delta_t(\bar{\omega} - l\Omega). \quad (6.6)$$

In particular, if the magnetic field has some azimuthal inhomogeneity we may estimate k according to the formula $k \approx (\Omega/\bar{\omega})(\Delta H/H)$, which is obtained from $\Delta_t(\bar{\omega} - l\Omega)/\bar{\omega} \simeq \Delta H/H$.*

* It may appear that the azimuthal inhomogeneity does not lead to a change of the frequencies $\bar{\omega}$ and Ω because the particle drift is perpendicular to ∇H. However, if one takes into account the motion of the particles along the lines of force, it is not difficult to see that H will change along the particle orbit, in fact in such a way that one has to take for $\Delta H/H$ the maximum value of the inhomogeneity along the line of force.

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As an example we give a comparison of the estimates obtained with the asymptotic formula (1.1).[7] We find the expression (2.3) for P and from (6.5) that $\Delta I/I \simeq \Delta^{(q)}(t\Omega)^{1/2}\pi^{-1/2}(2\omega)^{-(q+1)}$ or, in a time $\Delta t = \pi/(2\Omega)$, corresponding to the monotonic change of $\omega(t)$ in one direction that was assumed by KULSRUD,[7] $(\Delta I/I)_1 \simeq \Delta^{(q)}2^{-1/2}(2\omega)^{-(q+1)}$ which agrees with (1.1) in order of magnitude.

7. CRITERIA OF STOCHASTICITY

The equations (6.3)—(6.5) are based on the assumption of stochasticity, i.e. on the randomness of the phases ψ_{0l}. We now attempt to elucidate under what conditions this assumption may be justified.

For simplicity we discuss a periodic crossing of a single first-order resonance at equally spaced instants $T/2$. According to (6.2) we have $\Delta I \sim \sum_n \cos\psi_n$, where the ψ_n are the phases at the instants of resonance. If the frequencies $\bar{\omega}$ and Ω do not depend on I, we have

$$\psi_{n+1} - \psi_n = \psi_0 = 2.\int_{t_n}^{t_n+T/2}(\bar{\omega} - l\Omega)\,dt = \text{constant}$$

and therefore

$$\Delta I \sim \sum_n \cos n\psi_0 \leqslant 1/\sin\tfrac{1}{2}\psi_0. \qquad (7.1)$$

In this case ΔI is bounded and there is no stochasticity. We now take into account the non-linear nature of the oscillations. After each passage through resonance the frequency $\bar{\omega} - l\Omega$ will be changed by the quantity $(\Delta\omega)_n = (\Delta I)_n \times \partial(\bar{\omega} - l\Omega)/\partial I$ and in the next passage through resonance this will lead to an additional change of phase by the amount $\psi_{1n} = T(\Delta\omega)_n$. If $\psi \ll 1$ we come back to the previous case. If however, $\psi_1 \gg 1$ then ψ_1 also determines the phase change ψ_n. Unlike (7.1) this phase change will not be uniform but will depend on the previous phase ($\psi_{1n} \sim (\Delta\omega)_n \sim (\Delta I_n) \sim \cos\psi_n$) and a slight change of the previous phase will give rise to a change of 2π in the following one. It is evident that in this case the sequence of phases ψ_n will be near to random. However, a rigorous proof of this statement does not at present exist and we accept it as a hypothesis. The criterion of stochasticity then has the form

$$\nu_1 = \frac{2}{\sqrt{\pi}}\left|\frac{IP_l T\dfrac{\partial}{\partial I}(\bar{\omega} - l\Omega)}{\sqrt{\left|\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right|}}\right| \gg 1. \qquad (7.2)$$

Similar criteria have been obtained by GOWARD[12] and HINE[13] by means of numerical calculations on the motion of a non-linear oscillator under the action of short periodic impulses. The authors derive ν_1 as the criterion for instability. Its connexion with stochasticity is not examined. According to these papers instability begins at $\nu_1 = \pi$.

We shall give this criterion a somewhat different physical interpretation. If $\omega(x, t)$ is periodic in t then each harmonic in (3.5) and (3.9) will be modulated with a certain frequency $\Omega_d = 2\pi/T$ and can be expanded into new harmonic components of frequencies $2(\bar{\bar{\omega}} - l\bar{\Omega}) \pm p\Omega_d$ ($\bar{\bar{\omega}}$ denotes an average over phases varying with the frequencies Ω and Ω_d, and $\bar{\Omega}$ is averaged over phases varying with the frequency Ω_d). The coefficients of this expansion are $P_{lp} \simeq P_l(\Omega_d/(2\Delta_t(\bar{\omega} - l\Omega))^{1/2}$ (see Appendix, paragraph 1).

Inserting the last expression into (7.2) and putting

$$\left|\frac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right| \simeq \frac{\Delta_t(\bar{\omega} - l\Omega)}{2}\,\Omega_d,$$

we obtain

$$\nu_2 = 4\sqrt{\pi}\,\frac{\Omega_{ph}^{\,2}}{\Omega_d^{\,2}} \gg 1, \qquad (7.3)$$

where $\Omega_{ph}^{\,2} = \left|2IP_{lp}\dfrac{\partial}{\partial I}\left(\bar{\bar{\omega}} - l\bar{\Omega} \pm \dfrac{p\Omega_d}{2}\right)\right|$ is the square of the frequency of the phase oscillations for the new system of resonances (5.3) and Ω_d is the distance between them.

We apply the criterion (7.3) to the stationary case (see Section 5). Using (5.3) and taking into account that the distance between resonances is 2Ω we obtain

$$\kappa = \frac{\left|2\sqrt{\pi}\,IP_l\dfrac{\partial}{\partial I}(\bar{\omega} - l\Omega)\right|}{\Omega^2} \gg 1. \qquad (7.4)$$

If (7.4) is satisfied, the variation of I is stochastic.

Apparently this effect was observed by GARREN *et al.*[14] who used numerical methods to investigate the motion of a charged particle in an adiabatic trap. They discovered that near the forbidden cone there exists a region of 'unstable' orbits which escape from the trap stochastically. Let us apply criterion (7.4) to the data of this paper. We determine P_l from (3.5) in the same way as this was done in the example of Section 6.

$$(\Delta I/I)_{1/2} \simeq \pi P_l/\Omega,$$

where $(\Delta I/I)_{1/2}$ is the relative change of I in a half period (π/Ω) of the slow oscillation. This quantity was also evaluated by GARREN *et al.*[14] Putting $|\partial(\bar{\omega} - l\Omega)/\partial I| \simeq \bar{\omega}/I$, we find from (7.4) that instability begins at $(\Delta I/I)_{1/2} \geqslant 0{\cdot}1$, while from the data

of GARREN *et al.* it follows that $(\Delta I/I)_{1/2} \geqslant 0{\cdot}02$. The discrepancy should not be considered as very great, remembering that the estimates are very rough.

8. NON-STATIONARY CASE (SLOW PASSAGE THROUGH RESONANCE)

Evidently the condition for the slow passage through a resonance is $A \gg 1$ where A is defined in (6.1). As was shown by the author elsewhere[10] there are two basic regimes for the slow passage through a resonance: trapping and single passage. An important characteristic of the second regime is that ΔI is independent of the phase of the oscillations and, related to this, that the process is reversible even when the stochasticity condition (7.2) is satisfied. Therefore, if one and the same resonance is passed through periodically and slowly, in both directions, I will experience only small oscillations.

The trapping regime is characterized by the fact that when the frequencies $\bar{\omega}$ and Ω change I changes automatically owing to the explicit time dependence, in such a way that the condition of resonance $\bar{\omega} \approx l\Omega$ always remains satisfied. It is evident that if the frequencies $\bar{\omega}$ and Ω vary periodically the process will also be completely reversible.

Trapping may however exert an essential influence on the adiabatic processes when there is a considerable change of magnetic field with time, because the additional condition $\bar{\omega}/\Omega = \text{constant}$ must be satisfied. The main difficulty in utilizing this influence is connected with the fact that the region of trapping is small. Its width across each resonance is approximately equal to[1] $2\Delta_{ph}(\bar{\omega} - l\Omega) \sim \Omega_{ph}$ where Δ_{ph} is given by (5.2). If I has a uniform distribution among the particles and if therefore the frequencies are also uniformly distributed, the proportion of particles captured is Ω_{ph}/Ω. But this is just the ratio which must be small according to the stochastic criterion (7.4).

9. COMPARISON WITH EXPERIMENT

As far as we know, the only work in which quantitative data on the variation of the adiabatic invariant have been obtained is that of S. N. RODIONOV.[15] His comparison of experimental data with the theoretical results obtained above was performed for three magnetic field configurations (see Table 1). To find the Fourier coefficients Ω_{1m} the magnetic field on the axis of the system ($H(x) \sim \omega(x)$) was approximated by a function of the form $1/(x + b)^2$ where x is the distance from the turning point of the path ($\dot{x} = 0$) and b was so chosen as to obtain the best agreement. The accuracy of the approximation was 20–50 per cent. The following numerical values were assumed: mean distance of the particles from the axis of symmetry in the median plane of the magnetic field: $r_1 = 2$ cm; radius of curvature of the magnetic line of force $R = 2/n$ cm; index of the fall-off of magnetic field $n = (\gamma - 1)/640$; here $\gamma = H_m/H_0$, where H_m is the maximum field on the axis of the system (the magnetic mirror) and H_0 the minimum field on the axis.*

The variation of I was found from equation (6.5). I_0 was so chosen that $v_\perp \sim v_\parallel$. Initially (for $I \sim I_0$) the change of I was at its slowest both because of the higher order of the resonances passed through (large Ω, see (6.6)) and because of the smallness of P_l (smoother function $\dot{\omega}/\omega$ smaller Ω_{1m}). It was assumed that it was sufficient to determine the number N of reflections on the magnetic mirrors which led to a change $(\Delta I_{1/2})$ of I such that the order of the resonances

* For a more accurate description of the experiment see RODIONOV.[15]

TABLE 1.—COMPARISON OF THEORETICAL AND EXPERIMENTAL DATA

No.	γ	ρ/R	$\Delta H/H(\%)$	k	A	ν	κ	N(calc.)	N(exp.)
1	13	6×10^{-3}	<4	2·5	—	$< 2 \times 10^{-6}$	2×10^{-8}	∞	$>6 \times 10^6$
			3	3·5	—	$<2 \times 10^{-6}$	2×10^{-8}	∞	$>6 \times 10^6$
2	40	5×10^{-2}	<4	4	—	3×10^{-3}	0·4	∞	10^5
			9	2	10^{-2}	2	0·4	8×10^3	2×10^4
			15	1·5	2×10^{-2}	2	0·4	6×10^3	2×10^4
3	5	3×10^{-3}	<0·5	13·5	—	$<8 \times 10^{-6}$	4×10^{-7}	∞	$>3 \times 10^6$
			4	2	—	$<8 \times 10^{-6}$	4×10^{-7}	∞	$>3 \times 10^6$

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passed through decreased by a factor $\frac{1}{2}$,* because after this the rate of change of I increases. It was assumed that $\Delta I_{1/2}/I \simeq \frac{1}{2}k \ll 1$ where k is given by (6.6). Using the relation $dt = dN(\pi/\Omega)$ we obtain from (6.5)

$$N \approx \frac{\Omega^2}{2\pi^2 k^3 (P^{(k)})^2}. \tag{9.1}$$

The coefficient P was estimated according to the formula (1.1), the criteria A, ν and κ according to (6.1), (7.2) and (7.4) and it was assumed that

$$\left|\frac{\partial}{\partial I}(\bar{\omega} - l\Omega)\right| \simeq \bar{\omega}/I;$$

$$\left|\frac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right| \simeq \frac{\Delta_t(\bar{\omega} - l\Omega)}{2}\Omega_d \simeq \frac{1}{2}\left(\frac{\Delta H}{H}\right)\Omega_d\bar{\omega}.$$

The drift frequency of the particles $\Omega_d = v_d/r$ was estimated according to the formula[11]

$$\frac{\Omega_d}{\Omega} \simeq \frac{n}{2}(\rho_1/r_1)^2 \frac{\omega_1}{\Omega}\frac{v^2}{\overline{v_\perp^2}}\left(1 + \frac{\overline{v_\parallel^2}}{v^2}\right). \tag{9.2}$$

The results of the calculation are given in Table 1. The data of the last column are taken from RODIONOV.[15] In fact a comparison can only be performed for his field configuration[2].† For $\Delta H/H = 9$ and 15 per cent we find satisfactory agreement with experiment. A sharp discrepancy for $\Delta H/H < 4$ per cent can apparently be explained by the large value of κ (for absence of stochastic changes of I it is necessary that $\kappa \ll 1$. For $\kappa \gg 1$ the number of oscillations would be only about 1000).

10. CONCLUSION

The resonance exchange of energy among the degrees of freedom of a charged particle that has been considered in this paper certainly takes place in any magnetic trap. However, it represents a danger, (allowing particles to escape,) only in systems which have 'forbidden' directions for the particle velocities. In addition to systems with magnetic mirrors some systems with compensation of the toroidal drift[4] are in this class. The most unfavourable conditions from the point of view of resonance exchange exist in traps with 'corrugated' magnetic fields.[16] In such fields the amplitudes of the resonance harmonics of the slow oscillations (Ω_m) are considerably greater than in a monotonically changing field.

We note finally that similar resonance phenomena can occur in devices to contain plasma by means of high-frequency fields.[17,18] In such cases one has not an exchange of energy, but a change of energy due to the action of the high-frequency oscillations.

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APPENDIX

Estimate of the coefficients P and Q

1. We assume that $\Omega_{1m} \sim \Omega_{2m}$ and that their signs are random. We assume also that the F_n are all of the same order for $\omega_1/\Omega < n < \omega_2/\Omega$ and that $F_n = 0$

* Naturally in the general case one can have both $\frac{1}{2}$ or 1 according to the ratio of the quantities ρ/R and Q/Ω (see Section 4).

† Special experiments for the verification of the present theory were not performed because at the beginning of the calculations (autumn 1958) the experiments of S. N. RODIONOV were already completed and his apparatus dismantled.

outside this interval, while the signs of the F_n are random. ($\Delta\omega = \omega_2 - \omega_1$ is the range of variation of ω). Such a shape of spectrum is a good approximation in the case of harmonic frequency modulation and can also be utilized for estimates in other cases. Then, by Parseval's inequality $|F_n| \simeq (\Omega/\Delta\omega)^{1/2}$ and

$$|P_l| \simeq \frac{\sqrt{2}}{4}\sqrt{\frac{\Omega}{\Delta\omega}}\left(\sum_{m=\omega_1/\Omega}^{m=\omega_2/\Omega} {\Omega_{1m}}^2\right)^{1/2};$$

$$|Q_l| \simeq \sqrt{2}\,|P_l|\,. \tag{A1.1}$$

Applying Parseval's equality to the sum under the the root, we find that it does not exceed Ω^2 (see (3.4), $\dot{\omega}/\omega \sim \Omega$). Assuming also that $\Delta\omega \sim \bar{\omega}$ we obtain from (A1.1)

$$Q/\Omega \leqslant \sqrt{\Omega/\bar{\omega}}. \tag{A1.2}$$

2. If $\omega(t)$ has a discontinuity $\Delta^{(q)}$ in its qth derivative, so that its $(q+1)$th derivative contains a δ-function, direct calculation gives

$$\Omega_{1m} \to \frac{2}{\pi}\frac{\Omega\,\Delta^{(q)}}{\omega(2m\Omega)^q} \quad \text{for} \quad m \to \infty.$$

Inserting this into (A1.1) we get

$$P \simeq \frac{\sqrt{2}}{\pi}\frac{\Omega\,\Delta^{(q)}}{(2\omega)^{q+1}} \tag{A1.3}$$

It is easy to see that for (A1.3) to be valid, it is not at all necessary that there should be a mathematical discontinuity in $\omega(t)$. It is only necessary that (a) at a certain instant the derivative of $\omega(t)$ should change by $\Delta^{(q)}$ during a time $\ll 1/\omega$, and (b) at all other times it should only change during times $\gg 1/\omega$. This is just the condition that (A1.1) be applicable.

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A CYCLOTRON WITH A RADIALLY TRAVELLING MAGNETIC WAVE

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Abstract—Possible designs for a cyclotron with a radially travelling magnetic field wave are considered and relations between the parameters are deduced. One or more radially travelling waves can be produced in the magnet gap by using separate ring windings fed from an a.c. source. Two possible systems are proposed. In the first, the travelling wave is responsible for accelerating the particles; in the second, a travelling field is superimposed on the normal fixed field and the particles are accelerated in the resulting total field. A region of stability ($1 > n > 0$) can be obtained and this must move outwards at the radial velocity of the particles. Stability is possible if the absolute field strength increases sharply from the inside to the outside of the acceleration zone. In principle, the system would allow the production of particles with energies as high as desired. In spite of the cyclic nature of the action, the mean beam intensity could well be better than that from a synchrocyclotron because of better focusing. Some approximate calculations are presented for accelerators of this type designed to produce various energies. It is shown that for a given energy the weight and size can be substantially less than for other types of accelerator.

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1. GENERAL CONSIDERATIONS

THE proposed accelerator differs from a conventional cyclotron by virtue of the fact that the particles are accelerated in a circular magnetic field wave which travels radially outwards from the centre of the machine. Such a wave could be produced by the arrangement shown in Fig. 1. The diagram shows only one magnetic field wave but there could be several.

The wave velocity v_r must be such that the particles move in a region of the magnetic field where the condition $0 < n < 1$ is fulfilled throughout the acceleration period. That is, the radial component of the particle velocity, which is determined by the accelerating voltage, must be equal to the wave velocity of the field. The magnetic field can be increased in step with the moving wave as it travels outwards towards the maximum radius while at the same time maintaining the condition for vertical focusing. It will then be possible to keep the accelerating voltage constant in the region where the particles become relativistic and consequently such a cyclotron may be designed for as high an energy as one desires.

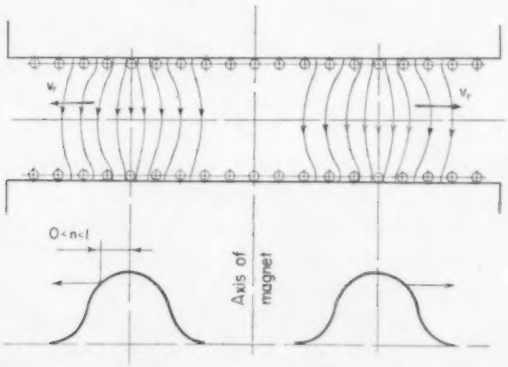

FIG. 1.—Arrangement for obtaining a radially travelling magnetic field wave using separate circular conductors on the surfaces of the magnet pole pieces.

It should be noted that in this case a value of n can be chosen which will give a larger instantaneous beam intensity compared with that available in an ordinary cyclotron or synchrocyclotron. Such an increase of beam intensity can be effected if, by choosing a sufficiently large gap width, a low magnetic field is created in the central region of the accelerator and if the gap width is made to fall off as the energy of the particles is increased. In spite of the fact that, as in the synchrocyclotron, the proposed machine accelerates the particles in bunches, one expects a larger mean beam intensity because of the better focusing conditions. In fact, one expects the mean beam intensity to be several orders of magnitude higher than in a synchrotron.

2. A PRACTICAL DESIGN FOR THE ELECTROMAGNET

A practical design for the electromagnet of the proposed accelerator is given in Fig. 2. The vacuum chamber is placed between two plates forming a gap with a conical profile. These plates are made of sheet iron with radial sections. They serve both as pole pieces and as part of the magnetic circuit. The plates

* Translated by D. L. ALLAN from *Atomnaya Energiya* **7**, 57 (1959).

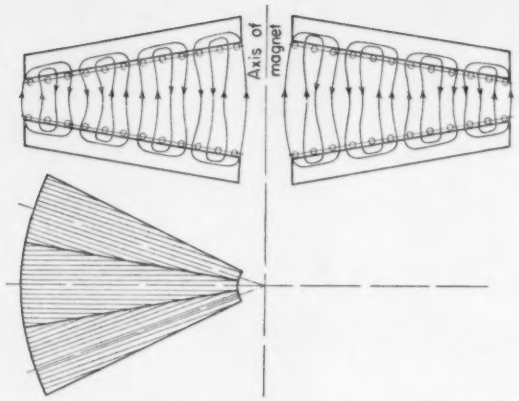

FIG. 2.—A possible design for the electromagnet. A few of the concentric magnetic field waves (multipolar system) are shown. Below: a design for the radial sections of the pole pieces.

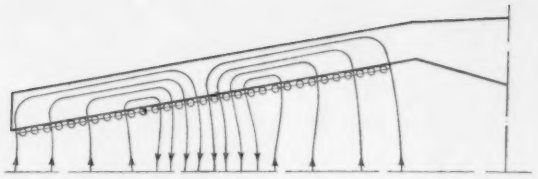

FIG. 3.—Version where the lines of force of the magnetic field wave cross the magnet gap twice.

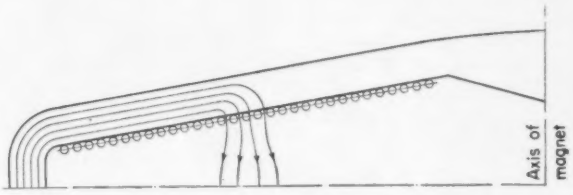

FIG. 4.—Version where the lines of force of the magnet field wave are closed through a special yoke around the outside of the pole pieces and cross the magnet gap only once.

consist of separate sectors, the sections of which are shown in Fig. 2. Concentric ring windings are fixed to the inside surfaces of the plates, facing each other.

There are several ways of supplying power to the coils. For example, they can be suitably grouped and fed from a three-phase supply. Then, as is known from the theory of electrical machines, it is quite easy to produce a magnetic field (Fig. 2) consisting of radially moving waves having a velocity which depends upon the frequency of the alternating current and upon the coil spacing. The lines of force are closed through the sections of the plates. Such an arrangement behaves as a multipolar system in which the weight of the magnetic circuit is substantially reduced because of the comparatively small thickness of the plates.

Another form of construction is shown in Fig. 3. Here there is only one magnetic field wave and the lines of force are closed either through the air gap of the machine (Fig. 3) or through a special yoke located outside the cylindrical surface of the accelerator (Fig. 4). One needs to use individual switching devices in the versions illustrated in Figs. 3 and 4 and it would seem that these would be best provided by ionic valves. To reduce the reactive power, the energy in the magnetic wave can be recovered when switching from one point to another.

The gap width in the proposed accelerator can be very small—it may be ~5–10 cm at maximum radius. The maximum field intensity will be at the final radius and the choice of value depends upon the magnetic properties of the iron and upon the feasibility of the required coil arrangement. For ordinary electromagnet steel, a suitable magnetic field intensity would be about 20,000 oersteds.

Since there is a limit to the velocity which can be given to the magnetic field wave, the energy gained by the particles per revolution must be of the order of hundreds or thousands of volts. Such voltages demand a rather high vacuum in the vacuum tank but this is offset by a considerably reduced high frequency generator power and by the fact that a rather simple accelerating system can be used.

In low energy accelerators, the accelerating system can consist of Dees but, for high energy machines, separate sector electrodes inserted between the sectors of the magnet plates must be used. The division of the magnet plates into sectors provides also a convenient means of access for the pumping lines.

Considerable difficulties arise in connexion with the design of a suitable injection system for the proposed accelerator. Here, we shall not be concerned with the details of how the particles are injected into the travelling magnetic wave field. For low and intermediate energies the central region can be designed as an ordinary cyclotron, as shown schematically in Fig. 6. In this diagram, the coil *A* provides the magnetic field for the initial cyclotron acceleration while the coils *B* excite the travelling magnetic field wave. It is possible to think of other methods of injection—in particular, one might use linear accelerators, synchrocyclotrons and synchrotrons. However, one design deserves special consideration. This is a cyclotron in which a travelling magnetic field wave is superimposed on a fixed magnetic field. In this case the magnetic circuit of the accelerator is the same as in a conventional cyclotron or synchrotron (Fig. 7). The pole pieces, which can form part of the vacuum chamber, are made of iron in sections. The weight of the electromagnet turns out to be much the same as in

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an ordinary cyclotron or synchrotron but, by using the travelling wave principle, a very high energy cyclotron can be built. In view of the special nature of this design we will consider it in some detail below.

3. RELATIONS BETWEEN THE FUNDAMENTAL PARAMETERS: APPROXIMATE CALCULATIONS FOR THE DESIGN WITHOUT A FIXED MAGNETIC FIELD

In order to have a constant particle revolution frequency and, consequently, a fixed accelerating voltage frequency, the magnetic field intensity at all radii must be

$$H = H_0(1 + W/E_0) \tag{1}$$

where H_0 is the field intensity at the start of the acceleration process, when the kinetic energy W is zero. If W_m is the energy at the final radius (the nominal energy of the accelerator) and H_m is the maximum field intensity at the final radius, then the initial field intensity is given by

$$H_0 = H_m/[1 + (W_m/E_0)]$$

For protons, the orbital radius (cm) for kinetic energy W (eV), rest energy E_0 (eV) and magnetic field H (oersteds) is:

$$r = \frac{\sqrt{[W(W + 2E_0)]}}{300H} \tag{2}$$

Inserting the value of H given by (1), we have

$$r = \frac{\sqrt{[W(W + 2E_0)]}}{300H_0[1 + (W/E_0)]}$$

$$\frac{dr}{dW} = \frac{E_0^3/300H_0}{(W + E_0)^2[W(W + 2E_0)]^{1/2}}.$$

Whence the step of the spiral is

$$\Delta_r = u\frac{dr}{dW} = \frac{uE_0^3/300H_0}{(W + E_0)^2[W(W + 2E_0)]^{1/2}}$$

where u is the energy acquired by a particle in one revolution.

The revolution period of the particles is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{c}[1 - \{E_0/(W + E_0)\}^2]^{-1/2}$$

where v is their velocity (cm sec^{-1}) and c is the velocity of light (cm sec^{-1}).

The particles have a radial velocity component:

$$v_r = \frac{\Delta r}{T} = \frac{uc}{600\pi H_0 r}\left(\frac{E_0}{W + E_0}\right)^3 \tag{4}$$

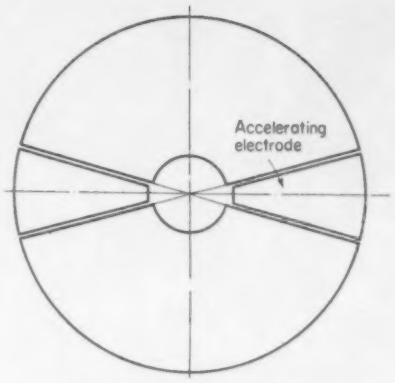

FIG. 5.—Accelerating electrode arrangement.

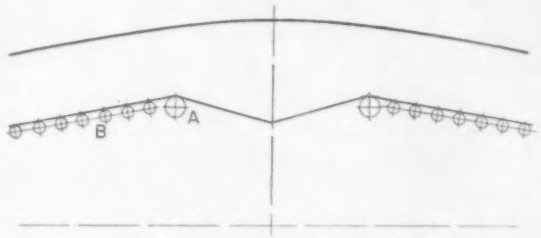

FIG. 6.—A possible injection scheme.
A—Coil for central injecting cyclotron.
B—Coils creating radially travelling magnetic field wave.

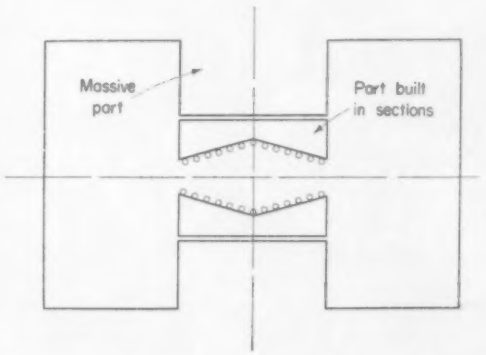

FIG. 7.—Design for a cyclotron in which the fixed magnetic field of an H-shaped electromagnet is superimposed upon a radially travelling magnetic field wave.

Equation (4) implies that the velocity of the magnetic wave in the central region of the accelerator must be appreciably greater than that in the final orbit.

Table 1 gives some of the parameters for such an accelerator for a number of energies. The following parameters were used in the calculations: $H_0 =$ 20,000 oersteds, gap width $\delta = 7{\cdot}5$ cm, frequency of current supplied to the magnetizing coils $f = 50$ c/s.

The weight of iron required was calculated assuming the pole piece plates were 15 cm thick and neglecting

TABLE 1

Final energy (MeV)	Final orbital radius (cm)	Field intensity at beginning of acceleration (oersteds)	Revolution frequency of particles (Mc/s)	Weight of iron (tons)	Power input (KVA)	
					Load of all coils, without switch gears	Load for two waves without energy recovery
100	74	18000	27·6	4	37000	—
1000	284	9750	14·5	60	550000	180000
10000	1810	1720	2·64	2400	—	750000
50000	8500	367	0·56	53000	—	—

the weight of the accessory installations which could substantially increase the weight of a large accelerator (the upper plate could be supported in its proper position by a composite bridge type of structure).

In working out the details of the construction it may be found that the required coil arrangement demands a larger gap between the poles and consequently more power than was assumed in the calculations. For the 100 MeV accelerator, the reactive power was calculated for the case where the excitor coils are supplied directly from the alternating current supply. The method of creating the travelling field using a three-phase alternating current supply is the same as that employed in ordinary electrical machines. The power requirement for the 1000 MeV accelerator was calculated for the direct supply from an alternating current system and for the case of a switching system in which the magnetic energy is not recovered from the individual sections of the excitor windings. The figures for the 10,000 MeV accelerator power requirements refer only to the case where the energy of the magnetic field is recovered. The power requirements of the highest energy accelerator (50,000 MeV) have not been examined, but such a machine would be feasible only if the magnetic energy is recovered. It is clear from the figures given in Table 1 that, from the point of view of the cost of such a project, it is doubtful if an accelerator of such a size would be worth building.

Values of the magnetic field intensity and magnetic wave velocity at various orbit radii for the 100 MeV cyclotron are given in Table 2 and the same parameters for the 1000 MeV cyclotron are given in Table 3. The accelerating voltage u for the 100 MeV cyclotron was taken as 115 V and for the 1000 MeV cyclotron as 1540 V. The radial magnetic field velocities and the corresponding accelerating voltages given in Tables 2 and 3 relate to the fixed accelerating voltage version. However, if the accelerating voltage is made to change during an acceleration cycle, it would be possible to obtain a constant or almost constant wave velocity for all orbital radii. This would reduce the weight of the excitor coils considerably.

TABLE 2

W (MeV)	H (oersteds)	r (cm)	V_r (m/sec)
10	18200	25·2	38·7
20	18400	39·3	24·2
40	18800	49·1	18·2
60	19150	53·3	14·1
80	19500	67·7	11·9
100	20000	74	10·1

TABLE 3

W (MeV)	H (oersteds)	r (cm)	V_r (m/sec)
15	9800	57·3	422
50	10200	101·5	214
100	10800	137·5	136
300	12900	209	54
500	14900	244	28·9
1000	20000	284	10

In order to simplify the calculation of the accelerating voltages, no allowance was made for the necessity of compensating for the e.m.f. induced by the rotational field. It is not difficult to show that the radially travelling magnetic field wave induces a rotational electric field whose vector is opposed to the accelerating voltage vector. To compensate for this, the accelerating voltage must actually be larger than the figures given above.

4. RELATIONS BETWEEN THE FUNDAMENTAL PARAMETERS: APPROXIMATE CALCULATIONS FOR THE FIXED MAGNETIC FIELD DESIGN

Equations (1) to (4) are valid also for the case where we have a radially travelling magnetic field

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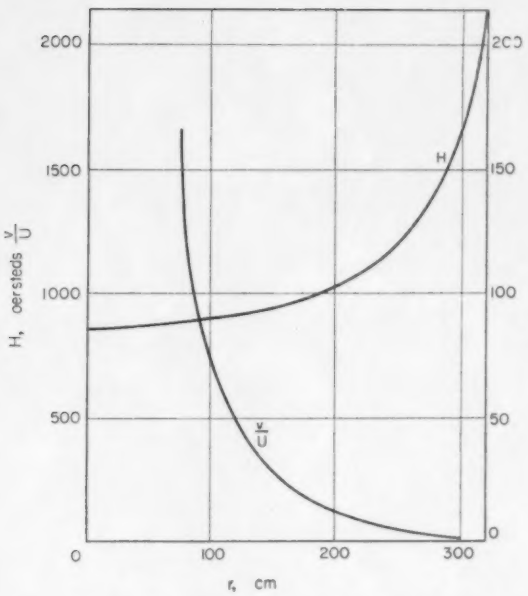

FIG. 8.—Curves giving magnetic field intensity and the ratio of the wave velocity to the accelerating voltage for a fixed field cyclotron with ~ 700 cm diameter pole pieces. Ordinate: H, oersteds.

wave superimposed upon a fixed magnetic field (see Fig. 7). As shown in this figure, the pole piece sections are constructed so that as the orbital radius of the accelerated particles increases there is a corresponding decrease in the width of the magnet gap. This reduction in gap width must be such that the magnetic field intensity varies with the particle energy according to equation (1). A curve giving the variation of the magnetic field intensity, calculated for a magnet with external pole diameter ~700 cm, is given in Fig. 8.

To maintain proper focusing conditions, a travelling wave must be superimposed upon the fixed field to provide an annular zone where the condition $0 < n < 1$ is fulfilled. A graph representing such a superposition of magnetic fields is given in Fig. 9. A current I passes through the set of circular conductors of width l located at radius r_1. Special switching gear moves the position of the set of coils being excited to a larger radius while at the same time the radius of curvature confining the magnetic field created by this set of conductors is correspondingly increased. The addition of the fixed and varying magnetic field forms a circular zone in which n has the required value.

If $(dH/dr)_1$ and $(dH/dr)_2$ are the absolute values of the derivative of the magnetic field intensity curve with respect to radius for the fixed flux and for the total field intensity in the annular zone, then the total current through the coil conductors in a section of the circular zone of width l is

$$I = 0{\cdot}8\delta_1 l\left[\left(\frac{dH}{dr}\right)_1 + \left(\frac{dH}{dr}\right)_2\right] \qquad (5)$$

where δ_1 is the gap width at radius r.

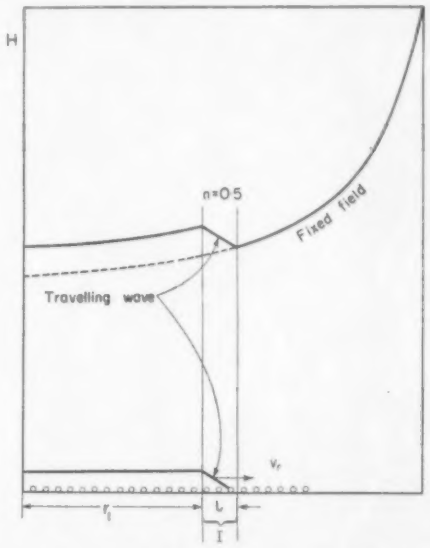

FIG. 9.—Diagram showing the superposition of a travelling magnetic wave upon the fixed field of a cyclotron.

TABLE 4

r (cm)	W (MeV)	H (oersteds)	$\left(\frac{dH}{dr}\right)_1$	$\left(\frac{dH}{dr}\right)_2$	v_r (m/sec)	u, b when v_r = const = 100 m/sec	δ (cm)	l (cm)	I (A)
0	0	8600	—	—	—	—	61·2	—	—
76	20	8750	—	—	595	437	—	—	—
113	50	9100	8·7	40	365	705	58	58	131000
155	100	9550	10	31	230	1130	55	—	—
213	200	10300	24·3	24·2	127	2050	51	—	—
276	500	13200	61	24·1	48·7	5340	39·8	39·8	107000
305	800	15900	108	26·2	24·9	10400	33	—	—
318	1000	17700	167	27·8	17·1	15200	29·7	—	—
327	1200	19600	250	30	12·5	20850	27	—	—
332	1350	21000	415	31·8	10	26100	25	25	224000

The e.m.f. induced by the travelling magnetic wave in a section of conductors of width l may be calculated from the following formula:

$$E = 2\pi r \frac{I}{0{\cdot}8\delta_1} \mathrm{v}_r \times 10^{-8} = 12{\cdot}3r \frac{I}{\delta_1} \mathrm{v}_r \times 10^{-8}\ \mathrm{V}.$$

The results of some approximate calculations for the case of a magnet with ~700 cm pole diameter are given in Table 4. The power required by the coils, assuming no energy recovery and a wave velocity of 100 m sec^{-1}, turns out to be 2–3 MW. The power is proportional to the frequency of the cycle and consequently also to the intensity.

The accelerating voltages given in Table 4 were computed for a constant wave velocity. In practice it is more convenient to keep the power required by the coils per cycle fixed and to decrease the wave velocity as the orbital radius increases. As in the preceeding calculations, the effect of the rotational electric field was neglected.

5. CONCLUSION

This account of the proposed accelerator has been limited to a discussion of the general principles of operation. Some possible designs for the construction of such a machine have been given as well as some approximate calculations. The results so far are encouraging and suggest that further calculations and the working out of constructional details would be profitable.

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A STEPWISE INCREASE IN THE ELECTRON CURRENT TO A PROBE PLACED IN A DISCHARGE IN A MAGNETIC FIELD*

A. V. ZHARINOV

(*Received* 13 *February* 1959)

Abstract—Preliminary results from a study of the effect of a magnetic field on electron diffusion in a plasma are described. A stepwise increase in the ratio of the electron current to the ion current to a probe has been found for a certain critical magnetic field strength. According to the preliminary results, this critical magnetic field is proportional to the gas pressure. These facts seem to indicate that there are two qualitatively different mechanisms of the transverse movement of electrons, one of these being diffusion by collisions.

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1. INTRODUCTION

ONE of the methods whereby the rate of motion of electrons in a plasma across a magnetic field can be estimated is to measure the ratio of the electron and ion currents to a probe in regions of the discharge where ionization may be neglected. On the basis of such measurements BOHM (GUTHRIE and WAKERLING[1]) has formulated a hypothesis that the electrons have an anomalously large transverse mobility. Although the mechanism of BOHM diffusion of electrons is not clear, it can be assumed that such diffusion does not always occur, and that in certain conditions the rate of ordinary diffusion of electrons by collision is sufficient to compensate the space charge of the ions.

In order to test this assumption, measurements have been made of the ratio of the electron current to a probe to the ion current for magnetic fields between 400 and 4500 oersteds.

2. APPARATUS AND MEASURING EQUIPMENT

Fig. 1 shows a diagram of the discharge apparatus. The discharge chamber is the cavity of a thick-walled copper cylinder *C*, whose ends are covered by a molybdenum anode *A* and a diaphragm *D*. A preheated cathode devised by B. N. Makov was used. The cathode itself was a tungsten cylinder *K*, 5 mm in diameter fixed on a special projection of the cathode unit *B* opposite an aperture of diameter 3 mm in the diaphragm. The cathode was heated by the thermionic current from a tungsten wire *N*. Six apertures of diameter 0·8, 1, 2, 3, 4 and 2 mm were made in the anode at 10 mm from its centre. Six collectors *S* were placed, one opposite each aperture, at ~0·5 mm from the rear face of the anode. These were used to find the probe characteristics. The collectors were attached to the screen *G* by means of glass insulators. The gas used was nitrogen, admitted to the discharge chamber by means of the tube *F*.

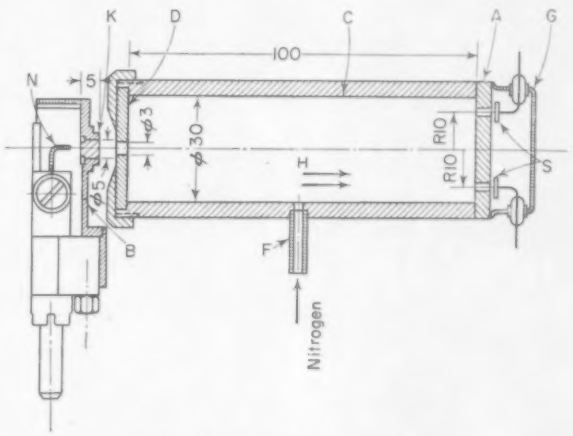

FIG. 1.—The discharge chamber. (Dimensions in millimetres.)

This apparatus was placed inside a metal vacuum chamber which was situated in the gap of an electromagnet and evacuated by an oil diffusion pump, which gave a vacuum of $\sim 10^{-5}$ mm mercury outside the discharge chamber with a continuous flow of nitrogen into it. The pressure inside the discharge chamber was about two orders of magnitude higher.

Fig. 2 shows the measuring equipment. A sawtooth oscillator *G* and a measuring resistance $R = 1\ k\Omega$ are connected in series between the probe *S* and the anode *A*. The constant component of the oscillator output voltage is balanced out by means of the battery *B* in such a way that the potential of the probe varies symmetrically with respect to the anode. From

* Translated by J. B. SYKES from *Atomnaya Energiya* **7**, 215 (1959).

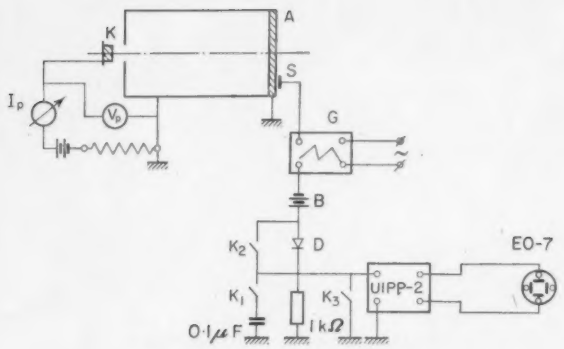

FIG. 2.—The measuring equipment.

the measuring resistance, the signal passes to the input of the d.c. amplifier UIPP-2, from whose output it is applied to the plates of the oscillograph EO-7.

To avoid linear distortion of the probe currents, which is considerable when the sensitivity is high, the probe potential is not measured directly, i.e. the oscillograph time-sweep generator is used to give the horizontal deflexion of the oscillograph beam.

In all measurements the amplitude of the probe potential relative to the anode was ~15 V. The measurements were made at a frequency of 50 cycles per second. With strong magnetic fields, marked oscillations always occurred in the discharge, and the electron part of the probe characteristic was strongly modulated by oscillations of a few tens of kilocycles per second. In such cases the measuring resistance was shunted by a capacity of 0·1 μF by using the switch K_1, and in this way the characteristic was averaged over the high-frequency components of the probe current.

When the electron current exceeded the ion current by a factor of more than 20–30, the ion current was measured at a higher amplification than the electron current, in order to record the ratio more accurately. By opening the switch K_2, the detector D was operated and the electron current was detected to avoid overloading the amplifier.

The switch K_3, which shunts the amplifier input, was used to apply the zero-current line when the probe characteristics were being photographed.

3. RESULTS OF THE MEASUREMENTS

Fig. 3 shows a series of oscillograms of probe characteristics corresponding to a discharge current of 1 A, a discharge potential of 180 V, and a pressure in the vacuum chamber of 7×10^{-5} mm mercury. The pressure in the discharge chamber was 2 to 3×10^{-3} mm mercury. These characteristics were obtained with the probe of diameter 3 mm, the remaining probes being earthed. The oscillograms

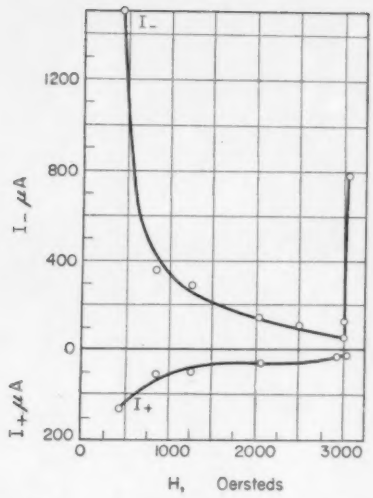

FIG. 4.—Electron current I_- and ion current I_+ at the probe as functions of the magnetic field strength. Probe diameter $D_S = 3$ mm, $I_p = 1$ A, $V_p = 180$ V, $p = 7 \times 10^{-5}$ mm mercury.

1–7 were taken without the capacitive shunting of the measuring resistance.

On examining these oscillograms, we observe the absence of oscillations, for both the ionic (lower branches) and the electronic (upper branches) currents when the magnetic field strength is less than 3000 oersteds. In a magnetic field only slightly exceeding this value, however (oscillogram 7), there are strong oscillations of the electron current at the probe. The oscillograms 1 to 6 show clearly a gradual decrease of the ratio of the electron and ion currents as the magnetic field increases.

The behaviour of the electron current I_- and the ion current I_+ at the probe, as functions of the magnetic field, is more easily seen from Fig. 4, which is based on oscillograms 1 to 8. The sudden stepwise increase of the electron current for a magnetic field ~3050 oersteds is of great interest. The ion current is almost unchanged at this point.

Fig. 5 shows the ratio I_-/I_+ as a function of the magnetic field strength. At first this ratio decreases from 11 at 450 oersteds to 4 at 2500 oersteds. Between 3000 and 3050 oersteds it rises from 5 to 17 if the electron current is measured from the lower envelope in oscillogram 7 (without capacitive shunt); if it is measured from the averaged oscillogram 8 (with capacitive shunt) the ratio I_-/I_+ is $\sim 10^2$ at a field of 3050 oersteds.

Figs. 6 and 7 show similar relations, with the current on an arbitrary scale, for the same probe but with a discharge current 0·6 A, a discharge potential 210 V, and the same pressure ($p = 7 \times 10^{-5}$ mm mercury). In this case the magnetic field was varied

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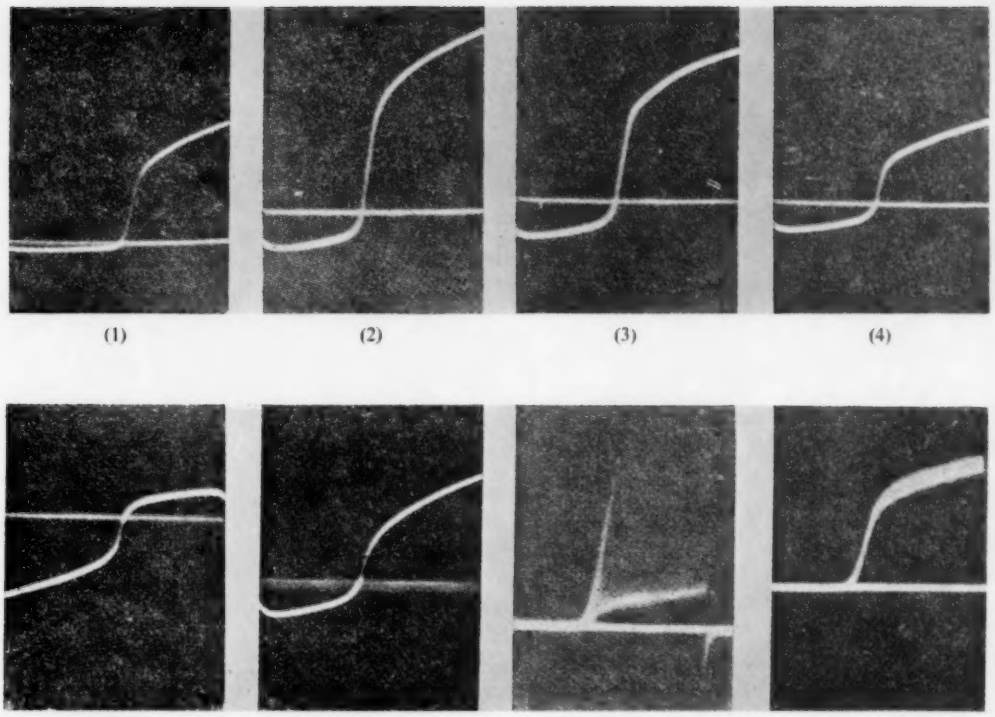

FIG. 3.—Oscillograms of probe characteristics of discharges at various magnetic field strengths H. k = amplification factor.

(1) $H = 430$ oersteds, $k = 50$
(2) $H = 850$ oersteds, $k = 400$
(3) $H = 1250$ oersteds, $k = 400$
(4) $H = 2050$ oersteds, $k = 400$
(5) $H = 2500$ oersteds, $k = 500$
(6) $H = 3000$ oersteds, $k = 1000$
(7) $H = 3050$ oersteds, $k = 100$
(8) $H = 3050$ oersteds, $k = 100$.

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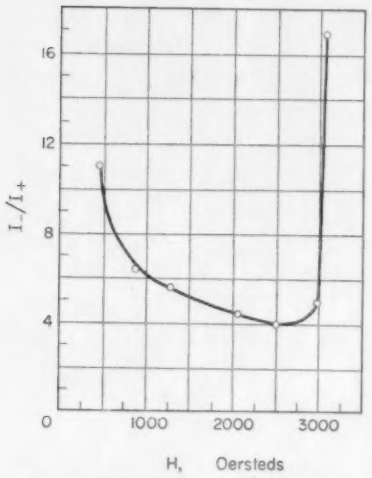

FIG. 5.—Ratio I_-/I_+ as a function of the magnetic field strength (from Figs. 3 and 4). $D_g = 3$ mm, $I_p = 1$ A, $V_p = 180$ V, $p = 7 \times 10^{-5}$ mm mercury.

over a wider range, and it is seen that the stepwise increase in the electron current is followed by a fairly smooth decrease.

When the pressure is halved ($3{\cdot}5 \times 10^{-5}$ mm mercury), the general nature of the probe characteristics is not greatly changed, but the discontinuity in the electron current, accompanied by strong oscillations, occurs at a magnetic field ~1500 oersteds, not ~3000 oersteds as when the pressure is 7×10^{-5} mm mercury.

4. DISCUSSION OF THE RESULTS

The principal experimental result of this work is the discovery of a stepwise increase, at a certain critical magnetic field, in the electron current to the probe, accompanied by strong oscillations. The critical value of the magnetic field increases proportionally to the gas pressure. This seems to indicate an important role of the quantities $\omega_+\tau_+$ and $\omega_-\tau_-$, which characterize the effect of the magnetic field on the plasma and may bring about a new state of the discharge with strong oscillations and a high velocity of the transverse motion of the electrons.

A rigorous quantitative analysis of the results is difficult for a variety of reasons. Firstly, the design of the probes used is imperfect, since the collecting surface is indefinite (both for electrons and for ions) on account of the appearance of a layer of negative anode potential fall at the edges of the probe aperture in the anode. Secondly, BOHM'S[1] theory of the collection of electrons by a probe in a discharge in a strong magnetic field is valid only for sufficiently long discharges, satisfying the condition

$$L > a/\sqrt{(D_-{}^p/D_-)},$$

where L is the length of the discharge, a the radius of the probe, $D_-{}^p$ the coefficient of transverse diffusion of electrons in a magnetic field, and D_- the electron diffusion coefficient without the magnetic field. In the present work, the reverse condition is more likely to be satisfied. Finally, in BOHM'S theory it is assumed that the kinetic energy of the ions in the plasma $eT_+ \ll eT_-$ (where T_- is the temperature of the electrons), and so the electron current should be observed to reach saturation at probe potentials which are $\sim T_+$ more positive than that of the plasma. The oscillograms in Fig. 3 show that no saturation of the electron current is observed in our experimental conditions. This seems to be due to the relatively high energy of the ions.[2]

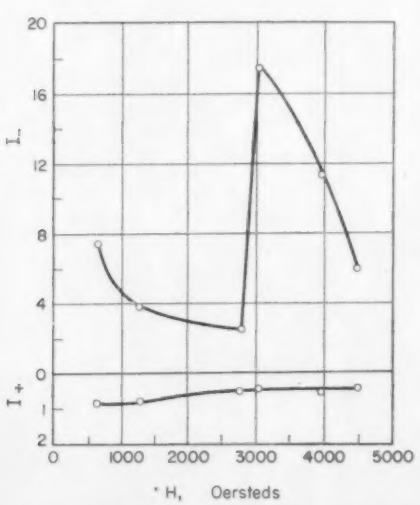

FIG. 6.—Electron current I_- and ion current I_+ at the probe as functions of the magnetic field strength. $D_g = 3$ mm, $I_p = 0{\cdot}6$ A, $V_p = 210$ V, $p = 7 \times 10^{-5}$ mm mercury.

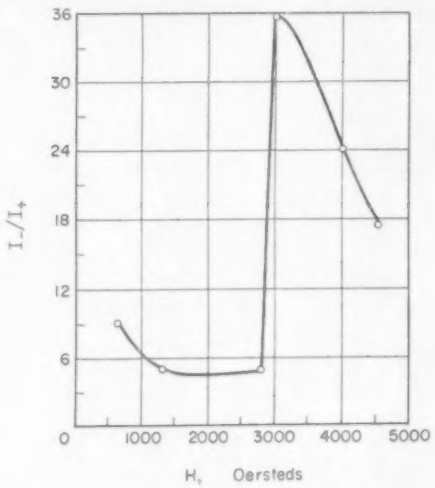

FIG. 7.—Ratio I_-/I_+ as a function of the magnetic field strength (from Fig. 6).

If we ignore all these objections and use BOHM's formula

$$\frac{I_-}{I_+} = \frac{8a\sqrt{(D_-D_-^{p})}(1 + T_+/T_-)}{0{\cdot}4 \times 2\pi a^2\sqrt{(2eT_-/M_+)}}$$

to estimate the ratio I_-/I_+, substituting $a = 0{\cdot}15$ cm, $\sqrt{M_+} \cong 4$, $T_- = 1$ eV, $1 + T_+/T_- \cong 1$, we find $I_-/I_+ \cong 3{\cdot}5$ for $H = 3000$ oersteds, and this is close to the measured value of $\sim$4. It should be noted that the experimental value of I_-/I_+ also cannot be determined accurately, on account of the non-saturation of the electron current.

Without placing too much reliance on this estimate, but bearing in mind the undoubted sudden increase in the electron current to the probe, we may conclude that these results, together with those of[3], confirm the existence of at least two qualitatively different mechanisms of the transverse motion of electrons. One mechanism, which operates at magnetic fields below the critical value, is, apparently, ordinary diffusion due to collisions. The other mechanism is still not clear, but it may be hoped that a more thorough and careful study of the observed critical transition in the state of the discharge will make possible an elucidation of the fundamental nature of this mechanism.

Acknowledgements—In conclusion, may I thank P. M. MOROZOV for assistance and interest in the work, and also S. SINOTOV, who helped with the measurements.

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THE DIFFUSION OF ELECTRONS IN A MAGNETIC FIELD*

A. V. ZHARINOV

(*Received* 13 *February* 1959)

Abstract—Experimental data concerning the anomalously high mobility of electrons across a magnetic field are discussed. It is shown that the concentration distribution of the secondary plasma of a hot-cathode discharge is almost independent of the transverse diffusion coefficient of the electrons, and so cannot be used to elucidate the mechanism of diffusion. The electron diffusion coefficient is estimated from the density of the electron current to the anode; the result confirms that the transverse mobility is anomalously high.

1. INTRODUCTION

THE study of various experimental data on a hot-cathode discharge in a strong longitudinal magnetic field led BOHM (GUTHRIE and WAKERLING[1]) to put forward the hypothesis that in such a discharge a new, hitherto unconsidered, mechanism of transverse motion of electrons exists, which is considerably more effective than diffusion by collisions.

In particular, one argument in favour of an anomalously high transverse mobility of electrons was furnished by an analysis of probe characteristics. Assuming that electron diffusion takes place by means of collisions, BOHM found a value of 4 or 5 for the ratio I_-/I_+ of the electron and ion currents to the probe, whereas the measured value of I_-/I_+ in corresponding conditions was from 20 to 35. Although BOHM's calculations lay no claim to great accuracy, it is nevertheless improbable that they should give too low a value of I_-/I_+, and so they must be given serious consideration.

Another experimental fact used by BOHM to demonstrate the anomalously high transverse mobility of electrons was the unexpectedly slow decrease in the plasma concentration with increasing distance from the primary electron beam. An approximate theoretical analysis was made of the concentration distribution in the secondary plasma (outside the primary electron beam) in the two-dimensional case. Using the condition of quasi-neutrality, assuming the transverse motion of ions and electrons to take place by diffusion and neglecting ionization in the secondary plasma, BOHM obtained the following expression for the distance x_0 over which the concentration decreases by a factor of e away from the primary beam:

$$x_0 = \sqrt{\frac{(T_+ + T_-)L}{(\beta T_+/D_+^{\,p}) + (\gamma T_-/D_-^{\,p})}}, \qquad (1)$$

where T_+ and T_- are the ion and electron temperatures in potential units, and $D_+^{\,p}$ and $D_-^{\,p}$ the ion and electron transverse diffusion coefficients. The coefficients β and γ relate the ion and electron current densities at the anode, $j_+^{\,a}$ and $j_-^{\,a}$, to the plasma concentration n:

$$\begin{aligned} j_+^{\,a} &= \beta n, \\ j_-^{\,a} &= \gamma n. \end{aligned} \qquad (2)$$

Assuming also that

$$\gamma T_-/D_-^{\,p} \ll \beta T_+/D_+^{\,p}, \qquad (3)$$

BOHM obtained, instead of (1), the expression

$$x_0 \approx \sqrt{\left[\left(1 + \frac{T_+}{T_-}\right)\frac{D_-^{\,p}L}{\gamma}\right]}. \qquad (4)$$

Thus, according to formula (4), a measurement of x_0 makes possible an estimate of the transverse diffusion coefficient of the electrons. Comparing the measured value of x_0 with that calculated from (4), and assuming that ordinary diffusion of electrons occurs, BOHM found a disagreement between theory and experiment: the electron diffusion coefficient calculated from (4) was two orders of magnitude greater than the diffusion coefficient resulting from collisions.

SIMON[2,3] has derived for x_0 an expression quite different from (4):

$$x_0 \approx \frac{L}{\pi}\sqrt{\frac{D_+^{\,p}}{D_{am.}}}, \qquad (5)$$

where $D_{am.}$ is the coefficient of longitudinal ambipolar diffusion. On comparing the measured values of x_0 with those calculated from formula (5), he obtained satisfactory agreement. It was also found that, as the magnetic field strength H increases, x_0 decreases as $1/H$. For this reason SIMON thinks that, in the experiments under consideration, the diffusion of both ions and electrons takes place by means of the familiar

* Translated by J. B. SYKES from *Atomnaya Energiya* 7, 220 (1959).

processes, and that there is no need to postulate the existence of the anomalously high transverse electron mobility deduced by BOHM.

However, SIMON does not use the results of the probe measurements already mentioned. Moreover, it is not really possible to determine the electron diffusion coefficient on the basis of a measurement of the distance x_0 in the discharge conditions considered by SIMON and by BOHM.

The object of the present paper is to show that, under certain conditions, the secondary plasma-concentration distribution does not depend on the value of the electron diffusion coefficient, but mainly on the motion of the ions. Consequently, it can be shown that BOHM and SIMON have incorrectly applied results concerning the plasma concentration distribution in attempting to elucidate the mechanism of the transverse diffusion of electrons.

2. THE PLASMA CONCENTRATION DISTRIBUTION IN A HOT-CATHODE DISCHARGE IN A LONGITUDINAL MAGNETIC FIELD

BOHM's formula (4) corresponds to ambipolar diffusion of the plasma, as was apparently first pointed out by SIMON. It is interesting to note that BOHM did not assume ambipolar diffusion as such, but his use of the condition (3) is equivalent to doing so.

However, in the discharge conditions considered (Fig. 1), the diffusion cannot be ambipolar. For, if ambipolar transverse diffusion takes place outside the primary beam, then everywhere outside that beam (P) the resultant longitudinal current to the anode will be zero, i.e.

$$\phi = T_- \log (v_-/v_+), \tag{6}$$

where ϕ is the potential of the plasma relative to the anode, and v_-, v_+ the mean longitudinal velocities of the electrons and ions. Formula (6) follows from the equality of the anode currents $j_+^a = j_-^a$. If the transverse diffusion is ambipolar, the potential ϕ must increase away from the primary beam by an amount of the order of T_+. This is incompatible with (6) for an equipotential anode, since the electron temperature T_- in a strong magnetic field must decrease quite rapidly outside the primary beam, on account of the cooling of the electrons by the gas.

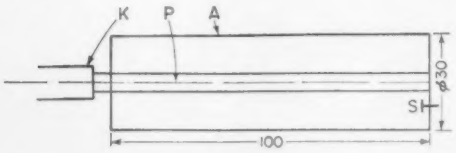

FIG. 1.—Diagram of the discharge chamber. (Dimensions in millimetres.)
K cathode; A anode; P primary beam; S probe.

This hypothesis may be experimentally confirmed by using the results of measuring the current to the anode in the secondary plasma, obtained by the author and shown in Figs. 2 and 3. These figures give the distribution of current over the surface of the anode, measured by a probe in the form of a disk of diameter 1 mm placed in the plane of the anode and moved perpendicular to the magnetic field. The primary beam had a diameter 5 mm. The curves shown correspond to a discharge potential of $V_p = 80$ V, a discharge current $I_p = 200$ mA, a magnetic field $H = 2300$ oersteds and a pressure of argon in the discharge chamber $p = 2 \times 10^{-3}$ mm mercury. The discharge gap was $L = 10$ cm, and the diameter of the discharge tube 30 mm.

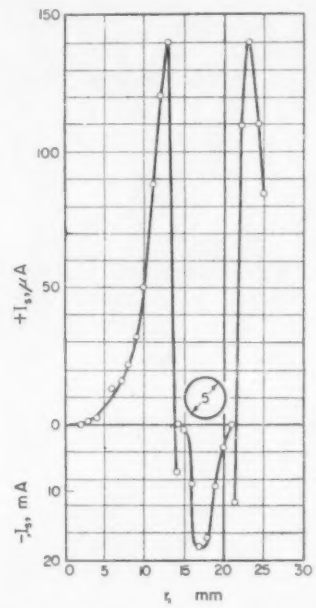

FIG. 2.—Distribution of current over the cross-section of the discharge. $I_p = 200$ mA, $V_p = 80$ V, $H = 2300$ oersteds, $p = 2 \times 10^{-3}$ mm mercury.

In Fig. 2 the distance from the probe to the wall of the discharge chamber is plotted as the abscissa; the ion current to the probe in microamperes is plotted upwards along the axis of ordinates, and the electron current in milliamperes is plotted downwards. The region occupied by the primary beam is shown by the circle. The probe is at a potential below that of the anode. The figure indicates that, at ~1 mm from the boundary of the primary beam, the ion current is

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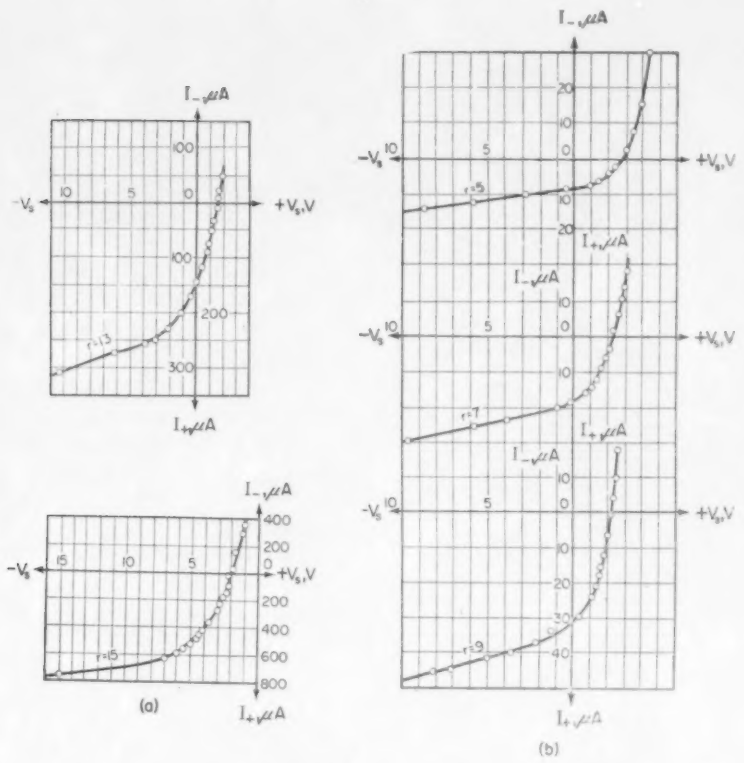

FIG. 3.—Ionic parts of probe characteristics at various points in the discharge cross-section.
(a) near the primary beam; (b) near the edge.

predominant. Furthermore, the current-voltage characteristics of the probe, obtained in the same experiment and shown in Fig. 3, indicate that the ion current to the probe considerably exceeds the electron current at distances of more than ~1 mm from the edge of the primary beam. At greater distances (Fig. 3b) the probe receives almost the saturation ion current. For example, at $r = 9$ mm, i.e. ~8 to 9 mm from the centre of the primary beam, the electron current is about 6 per cent of the ion current.

These results evidently prove that the diffusion in the secondary plasma is not ambipolar, and that the electrons move across the field considerably more slowly than the ions.

From the data thus obtained, we can deduce that the electrons in the secondary plasma are, as it were, in a closed potential vessel, and in a first approximation we can neglect those which escape, i.e. assume that the electron gas is in Boltzmann equilibrium, and hence there are almost no electron currents in the secondary plasma. Then it is easily shown that the characteristic distance x_0 is given by

$$x_0 = \frac{L}{\pi}\sqrt{\frac{D_+{}^p}{D_+}}. \qquad (7)$$

According to this formula, the distribution of concentration in the secondary plasma is independent of the electron diffusion coefficient; the physical significance of (7) is clear if we recall that, in diffusion of ions, the ratio of the mean square displacements across and along the magnetic field is

$$\overline{x_0{}^2}/(\tfrac{1}{2}L)^2 \approx D_+{}^p/D_+.$$

Formula (7) can also be derived from BOHM's theory if we assume that the opposite condition to (3) holds:

$$\gamma T_-/D_-{}^p \ll \beta T_+/D_+{}^p. \qquad (8)$$

This follows because the electron current to the anode is small, even in comparison with the ion current. From (8) we obtain, instead of (4),

$$x_0 = \sqrt{\left[\left(1 + \frac{T_-}{T_+}\right)\frac{D_+{}^p L}{\beta}\right]}. \qquad (9)$$

In longitudinal diffusion of ions, the coefficient β is in order of magnitude

$$\beta \cong \frac{D_+}{L}\left(1 + \frac{T_-}{T_+}\right). \qquad (10)$$

It is not difficult to see that, when (10) is substituted in (9), formula (7) is in fact obtained. It should be noted

that SIMON's formula (5) gives the same quantitative results as (7), but SIMON's formula seems less clear from the physical point of view, having been obtained by neglecting, in the charged-particle flux balance, terms proportional to the transverse mobilities of the ions and electrons. It is by no means evident that this neglect is justified.

When the distribution of concentration is practically independent of the electron diffusion coefficient, we may attempt to estimate the latter directly from the magnitude of the electron current to the anode at some point r. In order of magnitude this is

$$j_-{}^a \cong LD_-{}^p n/r^2. \tag{11}$$

The ratio of the electron and ion currents is

$$\frac{j_-{}^a}{j_+{}^a} \cong \frac{LD_-{}^p}{r^2 v_+}. \tag{12}$$

The lower probe characteristic in Fig. 3(b) ($r = 9$ mm) was obtained for a plasma concentration $\sim 10^{11}$ cm^{-3} and a neutral gas concentration $\sim 7 \times 10^{13}$ cm^{-3}. For this ratio of concentrations and $T_- \simeq 1$ eV, collisions between electrons and ions may be neglected. Then the electron diffusion coefficient in a magnetic field $H = 2300$ oersteds is $D_-{}^p \simeq 10$ cm^2 sec^{-1}. Assuming that the argon ions move freely[4] and have a longitudinal energy $\sim T_- = 1$ eV, we find $v_+ \simeq 2 \times 10^5$ cm sec^{-1}. Substituting these values and $L = 10$ cm, $r = 1$ cm in (12), we obtain

$$j_-{}^a/j_+{}^a \cong 5 \times 10^{-4}.$$

The measured ratio of currents at the anode is $\sim 6 \times 10^{-2}$ (see above). Although the estimate is very rough, the observed discrepancy of two orders of magnitude clearly does not support the theory that diffusion takes place by means of collisions.

A more convincing proof that there is some mechanism of transverse motion of electrons other than ordinary diffusion is furnished by the stepwise increase of the ratio of the electron current at the probe to the ion current,[5] and by the marked increase in the transverse diffusion coefficient.[6]

Acknowledgement—Thanks are due to I. V. KURCHATOV for his interest in the work and valuable criticism.

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LETTERS TO THE EDITOR

On 'superposed' accelerators*

(*Received* 7 *August* 1958)

IT has recently become obvious that it is necessary to study the interaction of electrons and photons of very high energy with material media.

A short time ago VEKSLER and the author indicated the possibility of using one and the same annular accelerator for alternative or simultaneous acceleration of different particles.[1] The conditions under which a 'superposed' accelerator works alternatively do not require any clarification, for, by simply adding to the existing injection and acceleration system of a synchrophasatron or an annular phasatron an injection and an acceleration system for electrons, one can switch to one or the other system alternatively and obtain acceleration of either electrons or protons.

The use of one and the same accelerator as a source of fast protons and electrons is extremely advantageous because such apparatus is complicated and costly.

Even more promising is the simultaneous combined acceleration of protons and electrons. (Evidently the simultaneous acceleration of positive ions of different masses should also be possible. With a great improvement of the vacuum one could accelerate simultaneously positive and negative ions.) Indeed if in future one could considerably increase the density of bunches of electrons and protons in an accelerating chamber (such an increase would be aided by the mutual attraction of oppositely charged and oppositely directed bunches) it would become possible to make two opposing bunches of electrons and protons collide with each other. In this case the velocity v of the reference system, in which the sum of the momenta of the two colliding particles is zero, is determined relative to the laboratory system by the equation

$$\frac{v}{c} = \frac{E_p\beta_p - E_e\beta_e}{E_p + E_e} = \frac{r_p - r_e}{r_p/\beta_p + r_e/\beta_e} \ll 1.$$

Here E_p and E_e respectively are the total energy of the protons and the electrons in the laboratory co-ordinate system and r_p, r_e are the radii of curvature of the proton and electron trajectories.

Thus, in practice, the laboratory system coincides with the centre of mass system of the colliding particles. This is equivalent to a considerable energy gain when compared to the situation when either a bunch of protons collides with a stationary electron target (the proton energy is increased by a factor k_p) or a bunch of electrons collides with a stationary proton target (the electron energy is increased by a factor k_e).

Here

$$k_p = \frac{1 + \beta_p\beta_e}{\sqrt{1 - \beta_e^2}}; \quad k_e = \frac{1 + \beta_p\beta_e}{\sqrt{1 - \beta_p^2}}.$$

To realize simultaneous superposed acceleration of bunches of particles of different masses one must, in practice, overcome a difficulty arising from the interaction of the proton bunch with the field resonator that accelerates the electrons. In the initial period of acceleration the protons revolve much more slowly than the electrons so that it is only necessary to avoid a loss of protons when their period of revolution T_p passes through integral multiples of the period of revolution T_e of the electrons. For annular accelerators we have

$$\frac{\omega_e}{\omega_p} = \frac{T_p}{T_e} \approx \frac{\beta_e}{\beta_p^2}.$$

The last passage through a 'resonance' of this kind takes place at an electron energy of only some hundred MeV (for not too high radio-frequency) while the radiative energy loss per revolution is still insignificant.

Further, at electron energies of the order of 3 BeV, the revolution frequencies of the electron and proton bunches will approach each other and a strong perturbation of the proton motion becomes possible. It is therefore necessary to find methods for reducing the influence of the electron resonator field on the proton motion. This field is very strong in order to compensate for radiation losses. The reaction of the proton accelerating element on the electron motion will evidently not be dangerous because in the initial stage of acceleration the frequency of the field accelerating the protons is considerably less than the frequency of revolution of the electrons and, at later times, when the frequencies approach each other, the voltage amplitude accelerating the protons becomes small compared to the voltage in the electron resonator.

We can indicate several methods of overcoming this difficulty. For a weak-focusing accelerator one can effect a spatial separation of the orbits and correspondingly of the limiting resonance frequencies of the electron and proton bunches and so diminish the perturbation of the proton orbits. Thus for instance with a frequency difference

$$\Delta\omega = \omega_e - \omega_p$$

the amplitude ΔR_p of the maximum radial perturbation of the proton bunch is

$$(\Delta R_p)_{\max.} \approx \frac{1}{\pi}\frac{R}{1-n}\frac{eV_e}{E_p}\frac{\omega_p}{\Delta\omega},$$

where R is the equilibrium radius, n the magnetic field index and V_e the voltage accelerating the electrons.

The above formula is valid at considerable values of the frequency spread $\Delta\omega/\omega_p$. If this spread becomes small, high-amplitude synchrotron oscillations begin to be excited which are connected with the protons going over to a new 'equilibrium' orbit corresponding to the orbital frequency ω_p. Another unfavourable circumstance is that in this transition the new 'equilibrium' orbit is situated inside the previous orbit and within the electron equilibrium orbit (since $\beta_p < \beta_e$) so that it is near the inner wall of the vacuum chamber.

The effect of the electron resonators on the motion of the protons can be reduced if they are excited at a frequency considerably higher than the orbital frequency of the electrons. However, in weak-focusing accelerators the choice of permissible frequency is limited by the effect of quantum fluctuations of

* Translated by N. KEMMER from *Atomnaya Energiya* **6**, 660 (1959).

the electron radiation on the amplitude of their synchrotron oscillations.[2,3] In strong-focusing accelerators such limitations on the permissible frequency are practically non-existent.

The most promising method appears to be to accelerate electrons by the electromagnetic field of a wave that progresses in the direction of the electron motion, or by using as accelerator a special system consisting of two or more resonators placed at different azimuth angles which have specially matched and controlled phase shifts. Let us assume that two resonators with equal voltage amplitudes V are excited at the orbital frequency of the electrons and have the adjustable phase shift

$$\Delta\phi = \phi_2 - \phi_1 = -\frac{\pi}{2}\left(1 - \frac{\Delta\omega}{\omega_p}\right).$$

Assume they are placed on two neighbouring rectilinear stretches of a race track, i.e. at an azimuth distance of $\theta = \pi/2$. The action of such a system of resonators on the electrons is equivalent to that of a single resonator of amplitude $2V$, while the system can neither accelerate nor decelerate the protons revolving in the opposite direction to the electrons. It is essential to regulate the phase shift only within narrow limits and sufficiently slowly in comparison with the slow change of orbital frequency of revolution for relativistic protons.

Simultaneous superposed acceleration of protons and electrons in a single accelerator is apparently possible at very high energies, up to several BeV.

E. M. Moroz

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The effect of a space charge on the motion of particles in accelerators*

(*Received* 20 *April* 1959)

In connexion with the problem of obtaining high fluxes in accelerators, the question of the effect of a space charge in the beam on the free oscillations of particles acquires considerable interest. This question has been discussed[1,2], but the screening effect of the chamber walls and of the iron in the electromagnet was not considered. In the present note both these effects are taken into account; only the horizontal walls of the chamber are considered, since the screening effect of the vertical walls is considerably less.

Let us suppose that the beam of particles is an infinite cylinder of radius ρ. Then, assuming the chamber walls to be perfect conductors and the magnetic permeability of iron to be infinite, we find that outside the beam the expressions for the vector and scalar potentials are [3]

$$A = -\pi Ne\beta\rho^2 \log\left[\cosh^2(\pi x/h)\sin^2(\pi z/h) + \sinh^2(\pi x/h)\cos^2(\pi z/h)\right], \quad (1)$$

$$\phi = \pi Ne\rho^2 \log\left[\{\cosh(\pi x/\alpha h) + \cos(\pi z/\alpha h)\}/\{\cosh(\pi x/\alpha h) - \cos(\pi z/\alpha h)\}\right],$$

where A is the component of the vector potential along the axis of the beam, $\beta = v/c$, N is the number of particles per unit volume, x and z are the deviations from the axis in the horizontal and vertical directions and α is the ratio of the vertical aperture of the chamber to the distance h between the poles of the electromagnet. Assuming that $\rho/h \ll 1$ and solving Maxwell's equations inside the beam, we find

$$\left.\begin{aligned} \mathscr{E}_x &= 2\pi eNx\left[1 - \frac{\pi^2}{6\alpha^2}(\rho/h)^2\right], \\ \mathscr{E}_z &= 2\pi eNz\left[1 + \frac{\pi^2}{6\alpha^2}(\rho/h)^2\right], \\ H_x &= 2\pi eN\beta z\left[1 + \frac{\pi^2}{3}(\rho/h)^2\right], \\ H_z &= 2\pi eN\beta x\left[1 - \frac{\pi^2}{3}(\rho/h)^2\right], \end{aligned}\right\} \quad (2)$$

where $\mathscr{E}_{x,z}$, $H_{x,z}$ are the electric and magnetic field components. When these fields are taken into account, we obtain the following equations for the free oscillations:

$$\left.\begin{aligned} x'' + k^2[1 - (n - \delta n_x)]x = 0, \\ z'' + k^2[n + \delta n_z]z = 0, \end{aligned}\right\} \quad (3)$$

where the differentiation is along the arc σ of the equilibrium orbit, k is the curvature of the orbit, n is the guiding magnetic field index of the accelerator, and

$$\delta n_{x,z} = -\frac{2\pi e^2 N}{k^2 E\beta^2}\left[(1 - \beta^2) \pm \frac{\pi^2}{6}(\rho/h)^2\left(\frac{1}{\alpha^2} - 2\beta^2\right)\right], \quad (4)$$

the upper and lower signs relating to δn_x and δn_z respectively, and E being the total energy of the particle.

From equations (3) we can easily derive the formula for the change in the frequency of free oscillations [4,5]:

$$\Delta\nu_{x,z} = \frac{Le^2N}{2E\beta^2}\left[(1 - \beta^2) \mp \frac{\pi^2}{6}(\rho/h)^2\left(\frac{1}{\alpha^2} - 2\beta^2\right)\right]\overline{|\Psi_{x,z}|^2}. \quad (5)$$

The quantity $\overline{|\Psi_{x,z}|^2}$ is the mean square modulus of the Floquet function, which in this case is normalized so that the Wronskian

$$\Psi_{x,z}\Psi^*{}'_{x,z} - \Psi^*_{x,z}\Psi'_{x,z} = -2i,$$

and L is the perimeter of the orbit. If the point considered is at the centre of a cell bounded by resonance bands, then, assuming that $|\Delta\nu_{x,z}| = \frac{1}{4}$ of the distance to the nearest resonance, we find the limitation on the beam current that can be injected into the accelerator chamber.

It follows from an analysis of formula (5) that the screening effect (the second term in the square brackets) gives a contribution of 10 to 20 per cent in the non-relativistic case. In the relativistic case ($\beta \sim 1$), however, this term may be of importance, leading to additional restrictions on the possible number of bunches.

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CURRENT PAPERS IN PLASMA PHYSICS, ACCELERATORS AND THERMONUCLEAR RESEARCH

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